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حل معادلة فولتيرا التكاملية التفاضلية من النوع الأول باستعمال التحويل الطبيعي المعدل

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الملخص:

تنشأ هذه المعادلات في سيناريوهات مثل عمليات تكوين الزجاج، ومشاكل الانتشار ، ونقل الإشعاع، ونمو الخلايا، ووصف حركة الأقمار الصناعية. سنقدم في هذا البحث التحويل الطبيعي المعدل كأداة لحل معادلات فولتيرا التكاملية التفاضلية من النوع الأول. طبقنا هذه الطريقة على بعض المسائل، مما يظهر فعاليتها ويوفر منهجية شاملة. تشير النتائج إلى أن التحويل الطبيعي المعدل فعال للغاية في معالجة معادلات فولتيرا التكاملية التفاضلية من النوع الأول.

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الكلمات المفتاحية: التحويل الطبيعي المعدل، مبرهنة الطي، معادلة فولتيرا التكاملية–التفاضلية.

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Solving the first-kind Convolution Type Volterra Integro-Differential Equation using the Modified Natural Transform

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Abstract

These equations arise in scenarios like glass formation processes, diffusion problems, radiation transfer, cell growth, and satellite motion description. This study introduces the Modified Natural Transform as a tool for solving Volterra integro-differential equations of the convolution type. The authors applied this method to solve numerical problems, demonstrating its efficacy and providing a comprehensive methodology. The results indicate that the Modified Natural Transform is highly effective for addressing convolution-type Volterra integro-differential equations of the first kind.

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1. Introduction

Integral transforms are important tools in diverse fields such as applied mathematics, theoretical mechanics, statistics, mathematical physics, and pharmacokinetics. These transforms provide precise analytical solutions, reducing the need for extensive calculations. Researchers like Aggarwal and others have employed various integral transformations (Mahgoub, Aboodh, Shehu, Elzaki, Mohand, Kamal) to obtain analytical solutions for first and second kind Volterra integral equations. Aggarwal et al. also presented solutions for second kind Volterra integro-differential equations by utilizing Mahgoub, Kamal, and Aboodh transformations. Aggarwal and colleagues utilized Mahgoub and Kamal transformations to solve linear partial integro-differential equations. They also applied Sawi, Mohand, Kamal, Shehu, Elzaki, Laplace, and Mahgoub transformations to address population growth and decay in mathematical models. Furthermore, they identified duality relations for various advanced integral transformations [1] and conducted comparative analyses of Mohand and other integral transformations [2]. In addition, Aggarwal et al. defined Elzaki, Aboodh, Shehu, Sumudu, Mohand, Kamal, Mahgoub, and Laplace transformations of error functions and their practical applications. They also presented solutions for ordinary differential equations with variable coefficients using the Mahgoub transform and delved into different integral transformations in subsequent works [3], particularly focusing on solving Abel's integral equations. Their research expanded to the study of Bessel's functions and the establishment of Mohand, Aboodh, Mahgoub, and Elzaki transformations [4]. Chaudhary et al. established connections between the Aboodh transform and other integral transforms. Aggarwal et al. [5] applied the Elzaki and Kamal transforms to solve linear Volterra integral equations of the first kind. Population growth and decay problems were addressed by Aggarwal et al. through the use of Aboodh and Sadik transformations. Aggarwal and Sharma [6] introduced the Sadik transform of the error function and its application in solving linear Volterra integro-differential equations of the second kind [7]. Aggarwal and Bhatnagar presented a solution for Abel's integral equation using the Sadik transform. Aggarwal conducted a comparative analysis of the Mohand and Mahgoub transforms. The Kamal transform of Bessel's functions was defined by Aggarwal [8]. Chauhan and Aggarwal tackled convolution-type linear Volterra integral equations of the second kind using Laplace transform. Sharma and Aggarwal [9] utilized Laplace transform to solve Abel's integral equation. Aggarwal and Sharma [10] introduced Laplace transform for solving a linear Volterra integral equation of the first kind. Mishra et al. clarified the relationship between the Sumudu transform and different integral transforms [11-13].

This paper focuses on determining the solution of a first-order convolution-type Volterra integro-differential equation by employing the Modified Natural Transform.

2. Definition of Modified Natural Transform

The Modified Natural transform of the function $f(x)$ for all $x \geq 0$ is defined as:

$$\mathrm{M}[f(x)] = \mathrm{M}(su,u) = \int_0^\infty f(sux)e^{-ux}dx = \frac{1}{su}\int_0^\infty f(x)e^{-\frac{x}{s}}dx,\ \ s > 0,\ \ u > 0 \qquad (1)$$

where $\mathrm{M}[\cdot]$ is the Modified Natural transform operator.

TABLE.1. Basic characteristics of modified natural transformations.

S.N.	Name of Property	Mathematical Form
1.	Linearity	$M\left[a f_1(x) + b f_2(x)\right] = a M\left[f_1(x)\right] + b M\left[f_2(x)\right]$
2.	Change of Scale	$M\left[f(\lambda x)\right] = \lambda M(\lambda su, u)$
3.	Shifting	$M\left[e^{ax} f(x)\right] = \frac{1}{1-as} M\left(\frac{1}{u(1-as)}\right)$
4.	First Derivative	$M\left[f'(x)\right] = \frac{M(su,u)}{s} - \frac{1}{su} f(0)$
5.	Second Derivative	$M\left[f''(x)\right] = \frac{M(su,u)}{s^2} - \frac{1}{s^2 u} f(0) - \frac{1}{su} f'(0)$
6.	nth Derivative	$M\left[f^{(n)}(x)\right] = \frac{M(su,u)}{s^n} - \sum_{i=0}^{n-1} \frac{1}{s^{i+1} u} f^{(n-1-i)}(0)$
7.	Convolution	$M\left[f(x) * g(x)\right] = su \cdot M_f(su,u) \cdot M_g(su,u)$

TABLE.2. Modified natural transformation of commonly encountered functions.

S.N.	$f(x)$	$M(su,u) = M[f(x)]$
1.	1	$\frac{1}{u}$
2.	x	$\frac{s}{u}$
3.	$x^{n-1}, \quad n = 1,2,3,\ldots$	$\frac{(n-1)!\, s^{n-1}}{u}$
4.	$\exp(ax)$	$\frac{1}{u(1-as)}$
5.	$\frac{x^{n-1} e^{ax}}{(n-1)!}, \; n = 1,2,\ldots$	$\frac{s^{n-1}}{u(1-as)^n}$
6.	$\frac{x^{n-1} e^{ax}}{\Gamma(n)}$	$\frac{s^{n-1}}{u(1-as)^n}$
7.	$\frac{\sin ax}{a}$	$\frac{s}{u(1+a^2 s^2)}$
8.	$\cos ax$	$\frac{1}{u(1+a^2 s^2)}$
9.	$\sinh ax$	$\frac{as}{u(1-a^2 s^2)}$
10.	$\cosh ax$	$\frac{1}{u(1-a^2 s^2)}$

3. Inverse Modified Natural Transform

If $f(x)$ is the inverse modified natural transform of $\mathrm{M}\left[f(x)\right] = \mathrm{M}(su,u)$, it is denoted as

$f(x) = \mathrm{M}^{-1}\left[\mathrm{M}(su,u)\right]$, where M^{-1} represents the inverse modified natural transform operator.

TABLE.2. Modified natural transformation of commonly encountered functions.

S.N.	$\mathrm{M}(su,u)$	$f(x) = \mathrm{M}^{-1}\left[\mathrm{M}(su,u)\right]$
1.	$\frac{1}{u}$	1
2.	$\frac{s}{u}$	x
3.	$\frac{(n-1)!s^{n-1}}{u}$	$x^{n-1}, \quad n = 1,2,3,\ldots$
4.	$\frac{1}{u(1-as)}$	$\exp(ax)$
5.	$\frac{s^{n-1}}{u(1-as)^n}$	$\frac{x^{n-1}e^{ax}}{(n-1)!}, \ n = 1,2,\ldots$
6.	$\frac{s^{n-1}}{u(1-as)^n}$	$\frac{x^{n-1}e^{ax}}{\Gamma(n)}$
7.	$\frac{s}{u(1+a^2s^2)}$	$\frac{\sin ax}{a}$
8.	$\frac{1}{u(1+a^2s^2)}$	$\cos ax$
9.	$\frac{as}{u(1-a^2s^2)}$	$\sinh ax$
10.	$\frac{1}{u(1-a^2s^2)}$	$\cosh ax$

4. Application of modified natural transform for solving first-kind Volterra integro-differential equations of convolution type

In this section, we solve a convolution-type Volterra integro-differential equation of the first kind using a modified natural transform.

A first-kind convolution-type Volterra integro-differential equation is expressed as:

$$\int_0^x K_1(x-u)f(u)\,du + \int_0^x K_2(x-u)f^{(n)}(u)\,du = g(x), \quad K_2(x-u) \neq 0 \qquad (2)$$

We denote

$$f(0)=g_0,\quad f'(0)=g_1,\quad f''(0)=g_2,\ \ldots\ ,\quad f^{(n-1)}(0)=g_{n-1} \tag{3}$$

Applying the modified natural transform to both sides of equation (2), we obtain

$$\mathrm{M}\left[\int_0^x K_1(x-u)f(u)\,du\right]+\mathrm{M}\left[\int_0^x K_2(x-u)f^{(n)}(u)\,du\right]=\mathrm{M}\left[g(x)\right] \tag{4}$$

By applying the convolution theorem of the modified natural transform to equation (4), we obtain

$$su\,\mathrm{M}\left[K_1(x)\right]\mathrm{M}(su,u)+su\,\mathrm{M}\left[K_2(x)\right]\mathrm{M}\left[f^{(n)}(x)\right]=\mathrm{M}\left[g(x)\right] \tag{5}$$

By applying the property of the "modified natural transform of the derivative of functions" to equation (5), we obtain

$$\begin{aligned}&su\,\mathrm{M}\left[K_1(x)\right]\mathrm{M}(su,u)\\&+su\,\mathrm{M}\left[K_2(x)\right]\mathrm{M}\left[\begin{array}{l}\dfrac{\mathrm{M}(su,u)}{s^n}-\dfrac{1}{su}f^{(n-1)}(0)-\dfrac{1}{s^2u}f^{(n-2)}(0)\\ \cdots-\dfrac{1}{s^{n-1}u}f'(0)-\dfrac{1}{s^nu}f(0)\end{array}\right]=\mathrm{M}\left[g(x)\right]\end{aligned} \tag{6}$$

Now, by utilizing equation (3) in equation (6), we obtain

$$\begin{aligned}&su\,\mathrm{M}\left[K_1(x)\right]\mathrm{M}(su,u)\\&+su\,\mathrm{M}\left[K_2(x)\right]\left[\frac{1}{s^n}\mathrm{M}(su,u)-\frac{g_0}{s^nu}-\frac{g_1}{s^{n-1}u}-\cdots-\frac{g_{n-2}}{s^2u}-\frac{g_{n-1}}{su}\right]=\mathrm{M}\left[g(x)\right]\end{aligned}$$

Thus,

$$\begin{aligned}&\left[su\,\mathrm{M}\left[K_1(x)\right]+\frac{u}{s^{n-1}}\mathrm{M}\left[K_2(x)\right]\right]\mathrm{M}(su,u)=\mathrm{M}\left[g(x)\right]+\\&\qquad\mathrm{M}\left[K_2(x)\right]\left[\frac{g_0}{s^{n-1}}+\frac{g_1}{s^{n-2}}+\cdots+\frac{g_{n-2}}{s}+g_{n-1}\right].\end{aligned}$$

Then,

$$\mathrm{M}(su,u)=\frac{\mathrm{M}\left[g(x)\right]+\mathrm{M}\left[K_2(x)\right]\left[\dfrac{g_0}{s^{n-1}}+\dfrac{g_1}{s^{n-2}}+\cdots+\dfrac{g_{n-2}}{s}+g_{n-1}\right]}{\left[su\,\mathrm{M}\left[K_1(x)\right]+\dfrac{u}{s^{n-1}}\mathrm{M}\left[K_2(x)\right]\right]}, \tag{7}$$

with $su\,\mathrm{M}\left[K_1(x)\right]+\dfrac{u}{s^{n-1}}\mathrm{M}\left[K_2(x)\right]\neq 0$

Applying the inverse modified natural transform to both sides of equation (7) yields the solution of the given Volterra integro-differential equation of the first kind in convolution form.

5. Numerical Problems

In this section, numerical problems are analyzed to elucidate the methodology in its entirety.

Example1. Examine the first-kind Volterra integro-differential equation of convolution type presented below

$$\int_0^x (x-u)\, f(u)\, du + \int_0^x (x-u)^2 f'(u)\, du = 3x - 3\sin x, \quad f(0) = 0. \tag{8}$$

By taking the modified natural transform of both sides of equation (8), we obtain

$$\mathbb{M}\left[\int_0^x (x-u)\, f(u)\, du\right] + \mathbb{M}\left[\int_0^x (x-u)^2 f'(u)\, du\right] = \mathbb{M}\left[3x - 3\sin x\right] \tag{9}$$

By applying the convolution theorem of the modified natural transform to equation (9), we obtain

$$su\mathbb{M}[x]\mathbb{M}[f(x)] + su\mathbb{M}[x^2]\mathbb{M}[f'(x)] = 3\mathbb{M}\left[x\right] - 3\mathbb{M}\left[\sin x\right]$$

thus,

$$\mathbb{M}[f(x)] + 2s\mathbb{M}[f'(x)] = \frac{3s}{u(1+s^2)}, \tag{10}$$

By applying the property of "modified natural transform of the derivative of functions" to equation (10), we obtain the following result.

$$\mathbb{M}(su,u) + 2s\left[\frac{\mathbb{M}(su,u)}{s} - \frac{1}{su} f(0)\right] = \frac{3s}{u(1+s^2)}, \tag{11}$$

Now, by utilizing the condition stated in equation (11), we obtain

$$\mathbb{M}(su,u) = \frac{s}{u(1+s^2)}, \tag{11}$$

By taking the inverse modified natural transform of equation (11), we obtain the solution of equation (8) as required

$$f(x) = \sin x.$$

Example2. Examine the first-kind Volterra integro-differential equation of convolution type presented below

$$\int_0^x \sin(x-u)\, f(u)\, du - \frac{1}{2}\int_0^x (x-u)\, f''(u)\, du = \frac{1}{2}x - \frac{x\cos x}{2},$$

$$f(0) = 0,\ f'(0) = 1. \tag{12}$$

By applying the modified natural transform to equation (12), we obtain

$$\mathbb{M}\left[\int_0^x \sin(x-u)\, f(u)\, du\right] - \frac{1}{2}\mathbb{M}\left[\int_0^x (x-u)\, f''(u)\, du\right] = \mathbb{M}\left[\frac{1}{2}x - \frac{x\cos x}{2}\right] \tag{13}$$

By applying the convolution theorem of the modified natural transform to equation (13), we obtain

$$\frac{s^2}{1+s^2}\mathbb{M}[f(x)] - \frac{1}{2}s^2\mathbb{M}[f''(x)] = \frac{s}{2u} - \frac{s^2}{2u(1+s^2)}. \tag{14}$$

By applying the property of the "modified natural transform of function derivatives" to equation (14), we obtain

$$\frac{s^2}{1+s^2}\mathrm{M}(su,u)-\frac{1}{2}s^2\left[\frac{\mathrm{M}(su,u)}{s^2}-\frac{1}{s^2u}f(0)-\frac{1}{su}f'(0)\right]=\frac{s}{2u}-\frac{s^2}{2u(1+s^2)}. \quad (15)$$

Now, utilizing the condition stated in equation (15), we obtain

$$\mathrm{M}(su,u)=\frac{s}{u(1+s^2)}. \quad (16)$$

By taking the inverse modified natural transform of equation (16), we obtain the solution of equation (12) as needed

$$f(x)=\sin x.$$

Example3. Examine the first-kind Volterra integro-differential equation of convolution type presented below

$$\int_0^x \cos(x-u)\,f(u)\,du+\int_0^x \sin(x-u)\,f'''(u)\,du=1+\sin x-\cos x,$$
$$f(0)=1,\ f'(0)=1,\ f''(0)=-1. \quad (17)$$

By applying the modified natural transform to both sides of equation (17), we obtain

$$\mathrm{M}\left[\int_0^x \cos(x-u)\,f(u)\,du\right]+\mathrm{M}\left[\int_0^x \sin(x-u)\,f'''(u)\,du\right]=\mathrm{M}\left[1+\sin x-\cos x\right] \quad (18)$$

By applying the convolution theorem of the modified natural transform to equation (18), we obtain

$$\mathrm{M}[f(x)]+\mathrm{M}[f'''(x)]=\frac{1}{u}. \quad (19)$$

By applying the property of the "modified natural transform of the derivative of functions" to equation (19), we obtain

$$\mathrm{M}(su,u)+\frac{\mathrm{M}(su,u)}{s^3}-\frac{1}{su}f''(0)-\frac{1}{s^2u}f'(0)-\frac{1}{s^3u}f(0)=\frac{1}{u}. \quad (20)$$

Now, applying the condition stated in equation (20), we obtain

$$\mathrm{M}(su,u)=\frac{s^3+s+1}{u(1+s^2)}=\frac{s}{u}+\frac{1}{u(1+s^2)}. \quad (21)$$

By taking the inverse modified natural transform of equation (21), we obtain the solution of equation (17) as needed

$$f(x)=x+\cos x.$$

Example4. Examine the first-kind Volterra integro-differential equation of convolution type presented below

$$\int_0^x (x-u)^2 f(u)\,du-\frac{1}{12}\int_0^x (x-u)^3 f'''(u)\,du=\frac{x^4}{12},\ f(0)=0,\ f'(0)=3,\ f''(0)=0. \quad (22)$$

By applying the modified natural transform to equation (22), we obtain

$$\mathrm{M}\left[\int_0^x (x-u)^2 f(u)\,du\right]-\frac{1}{12}\mathrm{M}\left[\int_0^x (x-u)^3 f''(u)\,du\right]=\mathrm{M}\left[\frac{x^4}{12}\right]. \tag{23}$$

By applying the convolution theorem of the modified natural transform to equation (23), we obtain

$$2\mathrm{M}[f(x)]-\frac{s}{2}\mathrm{M}[f''(x)]=\frac{2s}{u}. \tag{24}$$

By applying the property of the "modified natural transform of the derivative of functions" to equation (24), we obtain

$$2\mathrm{M}(su,u)-\frac{s}{2}\left[\frac{\mathrm{M}(su,u)}{s^3}-\frac{1}{su}f''(0)-\frac{1}{s^2u}f'(0)-\frac{1}{s^3u}f(0)\right]=\frac{2s}{u}. \tag{25}$$

Now, employing the condition stated in equation (25), we obtain

$$\mathrm{M}(su,u)=\frac{3s-4s^3}{u(1-4s^2)}=\frac{s}{u}+\frac{2s}{u(1-4s^2)}. \tag{26}$$

By taking the inverse modified natural transform of equation (26), we obtain the solution of equation (22) as needed

$$f(x)=x+\sinh 2x.$$

6. Conclusions

This paper effectively explores the use of modified natural transform in solving first-kind convolution-type Volterra integro-differential equations. By presenting four numerical problems, we demonstrate the utility of this transform. The results indicate that the modified natural transform is highly effective for addressing such equations. Furthermore, it is suggested that this transform could be applied to solve systems of first-kind convolution-type Volterra integro-differential equations in future studies.

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