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التحويل الطبيعي المعدل" تحويل تكاملي جديد، خصائصه وتطبيقاته

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الملخص:

في هذا البحث، نقدم تحويلًا تكامليًا جديدًا نسميه "التحويل الطبيعي المعدل"، يُمكن اعتبار هذا التحويل التكاملي كأساس لعدة تحويلات تكاملية جديدة محتملة. قمنا بإيجاد العديد من الخصائص الأساسية المرتبطة بهذا التحويل التكاملي الجديد في هذا البحث، على سبيل المثال، مبرهنة وجود التحويل الطبيعي المعدل، مبرهنة التحويل الطبيعي المعدل للمشتقات، مبرهنة الطي ومبرهنة التحويل الطبيعي المعدل العكسي. الميزة الرئيسية لهذا التحويل الجديد هي قدرته على حل المعادلات التفاضلية بمعاملات متغيرة وثابتة وبزمن أقل مقارنة مع باقي التحويلات الأخرى، من أجل توضيح كفاءة وفائدة التحويل الطبيعي المعدل المقدم لحل المعادلات التفاضلية، قمنا بتطبيق التحويل الجديد على عدة مسائل.

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الكلمات المفتاحية: التحويل الطبيعي المعدل، التحويل الطبيعي، تحويل لابلاس، مبرهنة الطي.

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A New Integral Transform “The Modified Natural Transform”, Its Properties and Applications

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Abstract

In this manuscript, we propose a new integral transform called the Modified Natural Transform. This transform can be considered as a base for several potential new integral transforms. Many fundamental properties about this new integral transform were created in this work. These properties include, for example, the existence theorem, the Modified Natural Transform of Derivatives theorem, convolution theorem, and inverse Modified Natural Transform. The main advantage of this new technique is its ability to solve differential equations with variable and constant coefficients. To demonstrate the efficiency and utility of the presented transform for solving differential equations, several examples are provided.

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1. Introduction

The differential equation holds significant relevance within the disciplines of physics, science, applied mathematics, chemistry, physiology, and engineering. Consequently, investigators propose the development of novel approaches for obtaining approximate solutions that are in close approximation to the exact solution.

In the field of mathematics, there exist numerous integral transforms that are frequently utilized to solve differential equations. As a result, there is a considerable body of literature dedicated to the study of integral transforms. One example of such a transform is the Laplace transform, which was first introduced by P.S. Laplace in the late 18th century [14]. The Laplace transform is not only the oldest integral transform but also the most widely utilized. Another integral transform is the Stieltjes transform, which was first introduced by T.S. Stieltjes in [18]. R.H. Mellin was the first who provided a systematic formulation of the Mellin transformation in [7]. The Hankel transform, also known as the Fourier-Bessel transform, was first developed by Hermann Hankel in [9]. D. Hilbert suggested the Hilbert transform in [5], and J. Radon founded the Radon transform in [10]. The Laguerre transform was introduced by Edmond Laguerre in [12]. Lastly, G.K.Watugula recently introduced Sumudu Transform in [6].

The natural transform was initially proposed by Z. H. Khan and W. A. Khan in [21], while the Elzaki transform was presented by Tarig M. Elzaki in [17]. Khalid S. Aboodh introduced the Aboodh transform in [11]. In [8], Srivastava suggested the new integral transform "M-transform". The ZZ transform was developed by Zafar in [20]. The Yang Transform was introduced by Xiao-Jun Yang in [19]. Mohand M. Mahgoub introduced the Mohand Transform in [13], and S. Ahmadi et al. proposed a new integral transform to solve higher order linear Laguerre and Hermite differential equations in [16]. Finally, R. Saadeh et al. introduced the ARA Transform in [15].

Yahya has a keen interest in integral transform methods. He utilized the Differential Transform to tackle boundary value problems represented by higher-order differential equations, as well as systems of differential equations. Furthermore, he proposed the combination of the homotopy perturbation method with the Sumudu Transform to address initial value problems for nonlinear partial differential equations. Additionally, he introduced the hybridization of the Natural Transform method with the homotopy perturbation method to solve the Van Der Pol Oscillator problem [1-4].

The aim of this paper is to exhibit the efficacy and adaptability of a novel integral transform, and to apply it to solve ordinary differential equations with both variable and constant coefficients. The subsequent sections are arranged in the following manner: Section 2 elucidates the fundamental concept of the modified natural transform, also, delves into the introduction of the modified natural transform for certain functions, and establishes certain properties. Section 3 showcases the application in solving ordinary differential equations, and Section 4 concludes the paper with a summary of the findings.

The modified natural transform was introduced by Yahya to facilitate the process of solving ordinary and partial differential equations in the time domain.

Typically, Fourier, Laplace, and Sumudu transforms are the convenient mathematical tools for solving differential equations. Additionally, the modified natural transform and some of its fundamental properties are used to solve differential equations.

A new transform called the modified natural transform we define for functions of exponential order. We consider functions in the set A , which is defined by:

$$\mathcal{A} = \left\{ f(x) \mid \exists \lambda_1, \lambda_2 > 0, \left| f(x) \right| < M \exp\left(\frac{|x|}{\lambda_i} \right), \text{ if } x \in (-1)^i \times [0, \infty) \right\} \qquad (1)$$

For a given function in the set A , the constant M is a real number, such as l_1, l_2. Here, l may be finite or infinite.

2. The Modified Natural Transform

Defintion1: The modified natural transform is denoted by the operator $\mathbb{M}(\cdot)$, which we define by the integral equations.

$$\mathbb{M}[f(x)] = \mathcal{M}(su, u) = \int_0^\infty f(sux) e^{-ux} dx = \frac{1}{su} \int_0^\infty f(x) e^{-\frac{x}{s}} dx, \; x \geq 0, \; s > 0, \; u > 0 \qquad (2)$$

The variables u and s in this transform are used to factor the variable t in the argument of the function f. This transform has a deeper connection with the Laplace transform. We also present many different properties of this new transform and the Sumudu transform, with a few properties extended.

The purpose of this study is to show the applicability of this interesting new transform and its efficiency in solving linear differential equations.

Theorem1: [Sufficient Condition for Existence of the modified natural transform]:

The modified natural transform $\mathbb{M}[f(x)]$ exists if it has exponential order and $\int_0^n |f(x)| dx$ exists for any $n > 0$.

Proof.

$$
\begin{aligned}
\frac{1}{su}\int_0^{\infty}\left|f(x)e^{-\frac{x}{s}}\right|dx &= \frac{1}{su}\int_0^{n}\left|f(x)e^{-\frac{x}{s}}\right|dx + \frac{1}{su}\int_n^{\infty}\left|f(x)e^{-\frac{x}{s}}\right|dx \\
&\leq \frac{1}{su}\int_0^{n}\left|f(x)\right|dx + \frac{1}{su}\int_0^{\infty}\left|f(x)\right|e^{-\frac{x}{s}}\,dx \\
&\leq \frac{1}{su}\int_0^{n}\left|f(x)\right|dx + M\frac{1}{su}\int_0^{\infty}e^{\frac{x}{\lambda}}\,e^{-\frac{x}{s}}\,dx \\
&= \frac{1}{su}\int_0^{n}\left|f(x)\right|dx + \frac{M}{su}\int_0^{\infty}e^{-\left(\frac{1}{s}-\frac{1}{\lambda}\right)x}\,dx \\
&= \frac{1}{su}\int_0^{n}\left|f(x)\right|dx + \frac{\frac{M}{su}}{-\left(\frac{1}{s}-\frac{1}{\lambda}\right)}\lim_{T\to+\infty}\left[e^{-\left(\frac{1}{s}-\frac{1}{\lambda}\right)x}\right]_0^T\ ;\ \lambda > s,\ u \neq 0 \\
&= \frac{1}{su}\int_0^{n}\left|f(x)\right|dx + \frac{\frac{M}{su}}{\frac{1}{s}-\frac{1}{\lambda}}
\end{aligned}
$$

The first integral exists and the second term is finite for $l > s$. Therefore, the integral $\frac{1}{su}\int_0^{\infty} f(x)e^{-\frac{x}{s}}dx$ converges absolutely and the modified natural transform $M[f(x)]$ also exists.

Theorem2: The duality relationship between the Laplace and the modified natural transforms.

Let $L[f(x)]$ be the Laplace transform and $M[f(x)]$ be the modified natural transform, then

i. $\mathcal{M}(su,u) = su\mathcal{L}\left(\frac{1}{s}\right)$

ii. $\mathcal{L}(s) = \frac{1}{su}\mathcal{M}\left(\frac{1}{s}\right)$

Proof.

$$\text{i. } \mathcal{M}(su,u) = \frac{1}{su}\int_0^{\infty} f(x)e^{-\frac{x}{s}}dx = su\left(\frac{1}{su}\int_0^{\infty} f(x)e^{-\frac{x}{1/s}}dx\right) = su\mathcal{L}\left(\frac{1}{s}\right).$$

$$\text{ii. } \mathcal{L}(s) = \int_0^{\infty} f(x)e^{-sx}dx = \frac{1}{su}\left(\int_0^{\infty} f(x)e^{-\left(\frac{1}{s}\right)x}dx\right) = \frac{1}{su}\int_0^{\infty} f(x)e^{-\frac{x}{s}}dx = \frac{1}{su}\mathcal{M}\left(\frac{1}{s}\right).$$

Lemma: The Modified Natural Transform of Some Functions

For any function $f(x)$, we assume that the integral equation (2) exists. The sufficient conditions for the existence of the modified natural transform are that $f(x)$ for x ³ 0 is piecewise continuous and of exponential order. Otherwise, the modified natural transform may or may not exist. In this section, we find the modified natural transform of simple functions.

1. Let $f(x) = 1$, then

$$\mathbb{M}[1] = \int_0^\infty e^{-ux} dx = \frac{1}{u}. \quad (3)$$

2. Let $f(x) = x$, then

$$\mathbb{M}[x] = \int_0^\infty suxe^{-ux} dx = \frac{s}{u}. \quad (4)$$

3. Let $f(x) = x^2$, then

$$\mathbb{M}[x^2] = \int_0^\infty s^2u^2x^2e^{-ux} dx = \frac{2s^2}{u}. \quad (5)$$

In general, if $n > 0$ is an integer number, then

$$\mathbb{M}[x^{n-1}] = \int_0^\infty s^{n-1}u^{n-1}x^{n-1}e^{-ux} dx = \frac{(n-1)!\,s^{n-1}}{u}. \quad (6)$$

The relationship can be written as the function of gamma in the

$$\mathbb{M}[x^{n-1}] = \frac{\Gamma(n)s^{n-1}}{u}. \quad (7)$$

4. Let $f(x) = e^{\alpha x}$, then

$$\mathbb{M}[e^{\alpha x}] = \int_0^\infty e^{\alpha sux}e^{-ux} dx = \frac{1}{u(1-\alpha s)}. \quad (8)$$

This outcome will prove beneficial in determining the modified natural transform for the following:

$$\mathbb{M}[\sin \alpha x] = \frac{\alpha s}{u(1+\alpha^2 s^2)}. \quad (9)$$

$$\mathbb{M}[\cos \alpha x] = \frac{1}{u(1+\alpha^2 s^2)}. \quad (10)$$

$$\mathbb{M}[\sinh \alpha x] = \frac{\alpha s}{u(1-\alpha^2 s^2)}. \quad (11)$$

$$\mathbb{M}[\cosh \alpha x] = \frac{1}{u(1-\alpha^2 s^2)}. \quad (12)$$

form

Lemma: The modified natural Transform of Derivatives

Let $\mathrm{M}(su, u)$ is the modified natural transform of $\mathrm{M}[f(x)]$. Then:

Let $\mathcal{M}(su,u)$ is the modified natural transform of $\mathbb{M}[f(x)]$. Then:

i. $\mathbb{M}\left[f'(x)\right]=\frac{\mathcal{M}(su,u)}{s}-\frac{1}{su}f(0)$.

ii. $\mathbb{M}\left[f''(x)\right]=\frac{\mathcal{M}(su,u)}{s^2}-\frac{1}{s^2u}f(0)-\frac{1}{su}f'(0)$.

iii. $\mathbb{M}\left[f^{(n)}(x)\right]=\frac{\mathcal{M}(su,u)}{s^n}-\sum_{i=0}^{n-1}\frac{1}{s^{i+1}u}f^{(n-1-i)}(0)$.

Proof.

i. $\mathbb{M}\left[f'(x)\right]=\int_0^\infty f'(sux)e^{-ux}dx$.

Integration by parts to find that:

$$\mathbb{M}\left[f'(x)\right]=\frac{\mathcal{M}(su,u)}{s}-\frac{1}{su}f(0) \qquad (13).$$

ii. $\mathbb{M}\left[f''(x)\right]=\int_0^\infty f''(sux)e^{-ux}dx$.

When using integration by parts and taking advantage of the previous relationship, we find:

$$\mathbb{M}\left[f''(x)\right]=\frac{\mathcal{M}(su,u)}{s^2}-\frac{1}{s^2u}f(0)-\frac{1}{su}f'(0). \qquad (14).$$

iii. We can prove the validity of the property using mathematical induction.

Theorem3: [Convolution Theorem]

Let $\mathbb{M}\left[f(x)\right]=\mathcal{M}_f(su,u)$, $\mathbb{M}\left[g(x)\right]=\mathcal{M}_g(su,u)$, then

$$\mathbb{M}\left[f(x)*g(x)\right]=su\cdot\mathcal{M}_f(su,u)\cdot\mathcal{M}_g(su,u). \qquad (25)$$

Proof.

We know that

$$f(x)*g(x)=\int_0^x f(\tau)g(x-\tau)d\tau$$

$$f(x)*g(x)=\int_0^x f(\tau)g(x-\tau)d\tau$$

Then

$$\begin{aligned}\mathbb{M}\left[f(x)*g(x)\right]&=\frac{1}{su}\int_0^\infty\left[f(x)*g(x)\right]e^{-\frac{x}{s}}\,dx\\&=\frac{1}{su}\int_0^\infty e^{-\frac{x}{s}}\,dx\int_\tau^\infty f(\tau)g(x-\tau)d\tau;\qquad 0\le\tau<x<\infty\\&=\frac{1}{su}\int_0^\infty f(\tau)d\tau\int_\tau^\infty g(x-\tau)e^{-\frac{x}{s}}\,dx\end{aligned}$$

When $x-\tau=v$ then $x=\tau+v$, we get

$$\begin{aligned}\mathbb{M}\left[f(x)*g(x)\right]&=\frac{1}{su}\int_0^\infty f(\tau)d\tau\int_0^\infty g(v)e^{-\frac{(\tau+v)}{s}}\,dv\\&=su\left(\frac{1}{su}\int_0^\infty f(\tau)e^{-\frac{\tau}{s}}d\tau\right)\left(\frac{1}{su}\int_0^\infty g(v)e^{-\frac{v}{s}}\,dv\right)\\&=su\cdot\mathcal{M}_f(su,u)\cdot\mathcal{M}_g(su,u)\end{aligned}$$

Theorem4: The modified natural Transform of $x^\alpha f^{(\beta)}(x);\ \alpha\ge1,\beta\ge0$

i. When $\alpha=1,\beta=0$, then

Proof.

We know that

$$f(x)*g(x)=\int_0^x f(\tau)g(x-\tau)d\tau$$

Then

$$\begin{aligned}\mathbb{M}\left[f(x)*g(x)\right]&=\frac{1}{su}\int_0^\infty\left[f(x)*g(x)\right]e^{-\frac{x}{s}}\,dx\\&=\frac{1}{su}\int_0^\infty e^{-\frac{x}{s}}\,dx\int_\tau^\infty f(\tau)g(x-\tau)d\tau;\qquad 0\le\tau<x<\infty\\&=\frac{1}{su}\int_0^\infty f(\tau)d\tau\int_\tau^\infty g(x-\tau)e^{-\frac{x}{s}}\,dx\end{aligned}$$

When $x-\tau=v$ then $x=\tau+v$, we get

$$\begin{aligned}\mathbb{M}\left[f(x)*g(x)\right]&=\frac{1}{su}\int_0^\infty f(\tau)d\tau\int_0^\infty g(v)e^{-\frac{(\tau+v)}{s}}\,dv\\&=su\left(\frac{1}{su}\int_0^\infty f(\tau)e^{-\frac{\tau}{s}}d\tau\right)\left(\frac{1}{su}\int_0^\infty g(v)e^{-\frac{v}{s}}\,dv\right)\\&=su\cdot\mathcal{M}_f(su,u)\cdot\mathcal{M}_g(su,u)\end{aligned}$$

Theorem4: The modified natural Transform of $x^\alpha f^{(\beta)}(x);\ \alpha\ge1,\beta\ge0$

i. When $\alpha=1,\beta=0$, then

$$\mathbb{M}[x\, f(x)] = s^2 \frac{d(\mathcal{M}(su,u))}{ds} + s\mathcal{M}(su,u). \tag{15}$$

ii. When $\alpha = 2, \beta = 0$, then

$$\mathbb{M}[x^2\, f(x)] = s^4 \frac{d^2}{ds^2}\mathcal{M}(su,u) + 4s^3 \frac{d}{ds}\mathcal{M}(su,u) + 2s^2\mathcal{M}(su,u). \tag{16}$$

iii. When $\alpha = 1, \beta = 1$, then

$$\mathbb{M}[xf'(x)] = s^2 \frac{d}{ds}\mathcal{M}(su,u). \tag{17}$$

With another formula,

$$\mathbb{M}[x\, f'(x)] = \mathcal{M}(su,u) - \frac{1}{u} f(0). \tag{18}$$

iv. When $\alpha = 2, \beta = 1$, then

$$\mathbb{M}[x^2\, f'(x)] = (4s - 2s^2)\frac{d}{ds}\mathcal{M}(su,u) + s^2 \frac{d^2}{ds^2}\mathcal{M}(su,u). \tag{19}$$

With another formula,

$$\mathbb{M}[x^2\, f'(x)] = 3s\mathcal{M}(su,u) - \frac{3s}{u} f(0) - \frac{s^2}{u} f'(0). \tag{20}$$

v. When $\alpha = 1, \beta = 2$, then

$$\mathbb{M}[xf''(x)] = \frac{d}{ds}\mathcal{M}(su,u) - \frac{1}{s}\mathcal{M}(su,u) - \frac{3}{su} f(0) - \frac{3}{su} f'(0). \tag{21}$$

vi. When $\alpha = 2, \beta = 2$, then

$$\mathbb{M}[x^2 f''(x)] = s^2 \frac{d^2}{ds^2}\mathcal{M}(su,u). \tag{22}$$

With another formula,

$$\mathbb{M}[x^2 f''(x)] = \mathcal{M}(su,u) - \frac{1}{u} f(0) - \frac{s}{u} f'(0). \tag{23}$$

Theorem5: Inverse Modified Natural Transform

Let $\mathbb{M}\left[f(x)\right] = \mathcal{M}(su,u)$ be the modified natural transform of $f(x)$, then

$$\mathbb{M}^{-1}\left[\mathcal{M}(su,u)\right] = f(x). \tag{24}$$

The modified natural transform Table for some functions.

No.	$f(x)$	$\mathrm{M}(su,u) = \mathrm{M}[f(x)]$
1	1	$\frac{1}{u}$
2	x	$\frac{s}{u}$
3	$x^{n-1}, \quad n = 1,2,3,\ldots$	$\frac{(n-1)!s^{n-1}}{u}$
4	$\exp(ax)$	$\frac{1}{u(1-as)}$
5	$\frac{x^{n-1}e^{ax}}{(n-1)!}, \; n = 1,2,\ldots$	$\frac{s^{n-1}}{u(1-as)^n}$
6	$\frac{x^{n-1}e^{ax}}{\Gamma(n)}$	$\frac{s^{n-1}}{u(1-as)^n}$
7	$\frac{\sin ax}{a}$	$\frac{s}{u(1+a^2s^2)}$
8	$\cos ax$	$\frac{1}{u(1+a^2s^2)}$
9	$\frac{e^{bx}\sin ax}{a}$	$\frac{s}{u((1-bs)^2+a^2s^2)}$
10	$e^{bx}\cos ax$	$\frac{1-bs}{u((1-bs)^2+a^2s^2)}$

3. Applications of the modified natural transform

The modified natural transform is used to solve differential equations, where the modified natural transform converts the linear differential equation with constant coefficients into an algebraic equation. The technique of solving algebraic equations is easier than solving initial value problems and higher-order linear differential equations with constant coefficients.

Example1. Use the modified natural transform to solve the given initial value problem:

$$y'(x) + y(x) = x + 1, \qquad y(0) = -1. \tag{26}$$

Solution. applying the modified natural transform to both sides, we obtain

$$\mathbb{M}[y'(x)] + \mathbb{M}[y(x)] = \mathbb{M}[x] + \mathbb{M}[1]$$
$$\frac{\mathcal{M}(su,u)}{s} - \frac{1}{su}f(0) + \mathcal{M}(su,u) = \frac{s}{u} + \frac{1}{u}$$

By satisfying the initial condition, we find that

$$\frac{\mathcal{M}(su,u)}{s} - \frac{1}{su}(-1) + \mathcal{M}(su,u) = \frac{s}{u} + \frac{1}{u}$$
$$(\frac{s+1}{s})\mathcal{M}(su,u) = \frac{s+1}{u} - \frac{1}{su}$$
$$\mathcal{M}(su,u) = \frac{s}{u} - \frac{1}{u(1+s)}$$

By applying the inverse modified natural transform to both sides, we have

$$\mathbb{M}^{-1}\Big[\mathcal{M}(su,u)\Big] = \mathbb{M}^{-1}\left[\frac{s}{u}\right] - \mathbb{M}^{-1}\left[\frac{1}{u(1+s)}\right]$$

Thus, the exact solution of the initial value problem (26) is given by

$$y(x) = x - e^{-x}$$

Example2. Use the modified natural transform to solve the given initial value problem:

$$y''(x) + y'(x) + y(x) = e^{-x}\sin x, \qquad y(0) = 2, \quad y'(0) = -1. \tag{27}$$

Solution. applying the modified natural transform to both sides, we obtain

$$\mathbb{M}[y''(x)] + \mathbb{M}[y'(x)] + \mathbb{M}[y(x)] = \mathbb{M}[e^{-x}\sin x]$$
$$\frac{\mathcal{M}(su,u)}{s^2} - \frac{1}{s^2u}f(0) - \frac{1}{su}f'(0) + \frac{\mathcal{M}(su,u)}{s} - \frac{1}{su}f(0) + \mathcal{M}(su,u) = \frac{s}{u((1+s)^2+s^2)}$$

By satisfying the initial condition, we find that

$$\frac{\mathcal{M}(su,u)}{s^2}-\frac{2}{s^2u}+\frac{1}{su}+\frac{\mathcal{M}(su,u)}{s}-\frac{2}{su}+\mathcal{M}(su,u)=\frac{s}{u((1+s)^2+s^2)}$$

$$\left(\frac{s^2+s+1}{s^2}\right)\mathcal{M}(su,u)=\frac{2+s}{s^2u}+\frac{s}{u((1+s)^2+s^2)}$$

$$\mathcal{M}(su,u)=\frac{1+s}{u((1+s)^2+s^2)}+\frac{1}{u}$$

By applying the inverse modified natural transform to both sides, we have

$$\mathbb{M}^{-1}\left[\mathcal{M}(su,u)\right]=\mathbb{M}^{-1}\left[\frac{1+s}{u((1+s)^2+s^2)}\right]+\mathbb{M}^{-1}\left[\frac{1}{u}\right]$$

Thus, the exact solution of the initial value problem (27) is given by

$$y(x)=e^{-x}\cos x+1$$

Example3. Use the modified natural transform to solve the given integro-differential equation:

$$x\,y'(x)+\int_0^t \exp(x-u)y(u)\,du=0,\qquad y(0)=1,\quad x\in\mathbb{R}_+. \tag{28}$$

Solution. applying the modified natural transform to both sides, we obtain

$$\mathbb{M}[xy'(x)]+\mathbb{M}\left[\int_0^x \exp(x-u)y(u)\,du\right]=0$$

$$\mathcal{M}(su,u)-\frac{1}{u}f(0)+su\cdot\frac{1}{u(1-s)}\cdot\mathcal{M}(su,u)=0$$

u

Solution. applying the modified natural transform to both sides, we obtain

$$\mathbb{M}[xy'(x)] + \mathbb{M}\left[\int_0^x \exp(x-u)y(u)\,du\right] = 0$$

$$\mathcal{M}(su,u) - \frac{1}{u}f(0) + su \cdot \frac{1}{u(1-s)} \cdot \mathcal{M}(su,u) = 0$$

By satisfying the initial condition, we find that

$$\mathcal{M}(su,u) - \frac{1}{u}(1) + su \cdot \frac{1}{u(1-s)} \cdot \mathcal{M}(su,u) = 0$$

$$\mathcal{M}(su,u)\left(\frac{1}{1-s}\right) = \frac{1}{u}$$

$$\mathcal{M}(su,u) = \frac{1}{u} - \frac{s}{u}$$

By applying the inverse modified natural transform to both sides, we have

$$\mathbb{M}^{-1}\left[\mathcal{M}(su,u)\right] = \mathbb{M}^{-1}\left[\frac{1}{u}\right] - \mathbb{M}^{-1}\left[\frac{s}{u}\right]$$

Thus, the exact solution of the integro-differential equation (28) is given by

$$y(x) = 1 - x$$

4. Conclusion

The primary objective of this study is to present the fundamental properties of the integral transform that we previously defined and called “The modified natural transform”. It has been found that the use of the modified natural transform is simpler than Natural transform, and the convolution theorem has also been proven for the modified natural transform. This new integral transform presents a novel mathematical approach for solving differential equations with both variable and constant coefficients, as well as their initial conditions. We recommend researchers and enthusiasts in this field to apply our new integral transform to fractional differential equations.

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