Weakly Generated Vector Spaces

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Abstract

It is important to appreciate at outset that the idea of a vector space in the algebraic abstraction and generalization of the Cartesian coordinate system introduced into the Euclidean plane, that is, a generalization of analytic geometry. Therefore, a number of interesting papers have been published on the concepts of generating sets and linearly independence. In this paper, we study the notion of weak generation of a vector space over a field and the notion of weakly independent sets as a generalization of linearly independent sets in vector spaces. We proved that if $\langle X \rangle_W$ is the subspace of V weakly generated by X , then $\langle X \rangle_W \subseteq \langle X \rangle$, and $X \subseteq \langle X \rangle_W$ if and only if $\langle X \rangle = \langle X \rangle_W$. Also, if $X \subseteq Y$ are subsets of V, then $\langle X \rangle_W \subseteq \langle Y \rangle_W$. If X is a finite subset of V and $0 \notin X$, then X is linearly independent if and only if $X \cup \{0\}$ is weakly independent. Also, we proved that the subset X of V is weakly

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independent if and only if each element $v \in \langle X \rangle_W$ can be written as a weak linear combination of X as the only form. Finally, interesting properties and corollaries are obtained for weakly independent subsets.

Key Words: Vector space, Generating and weakly generating, Linearly independent and weakly independent.

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الفضاءات المتجهية ضعيفة التوليد

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الملخص

تعد الفضاءات المتجهية من المفاهيم الجبرية المهمة ،وأهميتها لا تكمن في كونها بنية جبرية مجردة أو أنها تعميم لمفهوم الإحداثيات الديكارتية، بل لأنها تدرس المستوي الإقليدي كتعميم للهندسة التحليلية. لأجل ذلك ظهرت العديد من الدراسات والأعمال العلمية الممتعة تتناول مفهومي المجموعات المولدة والاستقلال الخطي. في هذه الورقة العلمية درسنا مفهوم التوليد الضعيف لفضاء متجهي فوق حقل ومفهوم الاستقلال الخطي الضعيف كتعميم لمفهوم الاستقلال الخطي في الفضاءات المتجهية. أثبتنا أنه إذا كان $\langle X \rangle_W$ الفضاء الجزئي المولد بضعف بالمجموعة X فإن $\langle X \rangle_W \subseteq \langle X \rangle$، وأن $X \subseteq \langle X \rangle_W$ عندما وفقط عندما $\langle X \rangle = \langle X \rangle_W$. وأنه إذا كانت X مجموعة جزئية منتهية من V وأن $0 \notin X$، عندئذ تكون المجموعة X مستقلة خطياً عندما وفقط عندما تكون المجموعة $X \cup \{0\}$ مستقلة خطياً بضعف. وأن المجموعة X من الفضاء V تكون مستقلة بضعف عندما وفقط

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عندما كل عنصر $v \in \langle X \rangle_W$ يمكن كتابته على شكل تركيب خطي ضعيف لعناصر المجموعة X بشكل وحيد. أخيراً تم الحصول على العديد من النتائج والخصائص الأخرى المتعلقة بالمجموعات المستقلة بضعف.

الكلمات المفتاحية: فضاء متجهي، التوليد والتوليد الضعيف، الاستقلال الخطي والاستقلال الخطي الضعيف.

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1. Introduction.

It is important to appreciate at outset that the idea of a vector space in the algebraic abstraction and generalization of the Cartesian coordinate system introduced into the Euclidean plane, that is, a generalization of analytic geometry. Therefore, a number of interesting papers have been published on the concepts of generating sets and linearly independence.

In 2014, Michal Hrbek [3] introduced the notion of weak independence as a generalization of independence, to modules over associative rings with an identity element, where a subset X of a left $R-$module M is called weakly independent if for any pairwise distinct elements $x_1, x_2, \cdots, x_n$ from X such that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$, then none of $\alpha_1, \alpha_2, \cdots, \alpha_n$ is invertible in R. Equivalent, a subset X of M is weakly independent if $x \notin Span(X \setminus \{x\})$, i.e., x is not in the submodule of M generated by $X \setminus \{x\}$. In addition, he studied a weak basis, where a weakly independent generating set X of a module M is called a weak basis. He proved that weakly independent generating sets are exactly generating sets minimal with respect to inclusion.

In 2016, Daniel Herden [2] studied another generalization of independence for modules as following, let M be an $R-$module and N be a submodule of M, a subset X of M is weakly independent over N provided that

$$x \notin N + Span(X \setminus \{x\})$$

for all $x \in X$. Also, a subset X of M is weakly independent if it is weakly independent over the zero submodule.

Weakly based Abelian groups were studied in [4] and [5]. In [4], the authors obtained their full characterization in terms of dimensions of certain residual vector spaces.

In this paper, we study the notion of weak generation of a vector space over a field and the notion of weakly independent sets as a generalization of linearly independent sets in vector spaces.

In section 2, we study the notion of weakly generating set, where a subset $\{v_1, v_2, \cdots, v_n\}$ of a vector space V over a field F is weakly generating of V if for every $v \in V$ there exist $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ such that $v = \sum_{i=1}^{n} \alpha_i v_i$ and $\sum_{i=1}^{n} \alpha_i = 0$. We proved that if $\langle X \rangle_W$ the subspace of V weakly generated by X, then $\langle X \rangle_W \subseteq \langle X \rangle$ and $X \subseteq \langle X \rangle_W$ if and only if $\langle X \rangle = \langle X \rangle_W$. Also, if $X \subseteq Y$ are subsets of V, then $\langle X \rangle_W \subseteq \langle Y \rangle_W$.

In section 3, we study the notion of weakly independent sets, where a subset $\{v_1, v_2, \cdots, v_n\}$ of a vector space V over a field F is weakly independent if for every $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ such that $\sum_{i=1}^{n} \alpha_i v_i = 0$ and $\sum_{i=1}^{n} \alpha_i = 0$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. We proved that if X is a subset of V and $0 \notin X$, then X is linearly independent if and only if $X \cup \{0\}$ is weakly independent. Also, we proved that the subset X of V is weakly independent if and only if each element $v \in \langle X \rangle_W$ can be written as a weak linear combination of X as the only form. Let X be a weakly independent and non-independent subset of V, if there exists an element $v \in X$ that can be written as a linear combination of $X \setminus \{v\}$, then $X \setminus \{v\}$ is independent. Also, it is proved that if X is a subset of V and $0 \notin X$, then X is maximal independent if and only if $X \cup \{0\}$ is maximal weakly independent, and if X is maximal weakly independent , then $V = \langle X \rangle_W$. Finally, interesting properties and corollaries are obtained for weakly independent subsets.

Throughout this paper, all vector spaces V are left over a field F as in [1], a finite subset X of a vector space V over F is called a basis of V [7], if it is generated of V and linearly independent. If V is a finite-dimensional vector space and X is a finite subset of V, then X is a basis of V if and only if X is maximal linearly independent if and only if X is minimal generating of V [6].

2. Weak generation of vector spaces.
In this section, we study a special case of linear combinations of a finite subset of a vector space over a field. We start with the following definition:
Definition. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. We say that every linear combination of X has the form $\sum_{i=1}^{n} \alpha_i v_i$ where $\alpha_i \in F$ for every $1 \le i \le n$ such that $\sum_{i=1}^{n} \alpha_i = 0$ is a *weak linear combination* of X. If there exist elements $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ for $v \in V$ such that $\sum_{i=1}^{n} \alpha_i = 0$ and $v = \sum_{i=1}^{n} \alpha_i v_i$, then we say that v is *written as a weak linear combination* of X.
Corollary 2.1. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. With $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0 \in F$, we notice that $\sum_{i=1}^{n} \alpha_i = 0$ and $\sum_{i=1}^{n} \alpha_i v_i = 0$, i.e., the zero element of V can be written as a weak linear combination of any finite subset X of V.
Lemma 2.2. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. Then the set of all weak linear combinations of X:
$\langle X \rangle_W = \{ \sum_{i=1}^{n} \alpha_i v_i : \alpha_i \in F, \forall\, 1 \le i \le n \;\; \wedge \; \sum_{i=1}^{n} \alpha_i = 0 \}$
is a subspace of V.
Proof. Since $0 \in \langle X \rangle_W$ where $\sum_{i=1}^{n} 0\, v_i = 0$, then the subset $\langle X \rangle_W$ is nonempty. Let $u, v \in \langle X \rangle_W$ and $\lambda \in F$, then there exist elements $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ such that $u = \sum_{i=1}^{n} \alpha_i v_i$ and $\sum_{i=1}^{n} \alpha_i = 0$, and elements $\beta_1, \beta_2, \cdots, \beta_n \in F$ such that $v = \sum_{i=1}^{n} \beta_i v_i$ and $\sum_{i=1}^{n} \beta_i = 0$. It is clear that $\lambda u + v = \sum_{i=1}^{n} (\lambda \alpha_i + \beta_i)\, v_i$ and

$\sum_{i=1}^{n}(\lambda\alpha_i+\beta_i)=0$. Therefore, $\lambda u+v\in\langle X\rangle_W$, i.e., the set $\langle X\rangle_W$ is a subspace of V.

According to the last lemma, we can form the following definition:

Definition. Let V be a vector space over a field F and $X=\{v_1,v_2,\cdots,v_n\}$ be a subset of V. We call the subspace:

$V'=\langle X\rangle_W=\{\sum_{i=1}^{n}\alpha_i v_i:\alpha_i\in F,\forall\ 1\le i\le n\ \wedge\ \sum_{i=1}^{n}\alpha_i=0\}$

a *weakly generated subspace* by X. If there exists a finite subset Z of V such that $V=\langle Z\rangle_W$, then we say that V is a *weakly finite generated space*, i.e., any element $v\in V$ is written as a weak linear combination of Z.

Example. With R as the field of real numbers, let $X=\{(1,0),(0,1),(2,3)\}$ be a subset of the vector space R^2 over R. It is easy to show that any $(x,y)\in R^2$ is written as a weak linear combination of X by the form:

$$(x,y)=\frac{x-y}{2}(1,0)+\frac{y-3x}{4}(0,1)+\frac{x+y}{4}(2,3).$$

Thus, R^2 over R is a weakly finite generated space by X. We note that $X\subset R^2=\langle X\rangle_W$.

Lemma 2.3. Let V be a vector space over a field F. The following hold:

1 – $\langle v\rangle_W=\{0\}$ for every element $v\in V$.

2 – If $\langle X\rangle_W\neq\{0\}$, then $Card\ X\ge 2$ for any finite subset X of V.

Proof. Obvious.

The following lemma shows the relationship between $\langle X\rangle_W$ and $\langle X\rangle$, where X is a finite subset of a vector space V over a field F.

Lemma 2.4. Let V be a vector space over a field F and $X=\{v_1,v_2,\cdots,v_n\}$ be a subset of V. Then $\langle X\rangle_W\subseteq\langle X\rangle$.

Proof. Let $v \in \langle X \rangle_W$, then there exist elements $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ such that $v = \sum_{i=1}^{n} \alpha_i v_i$ and $\sum_{i=1}^{n} \alpha_i = 0$. Thus, v is written as a linear combination of X, i.e., $v \in \langle X \rangle$, then $\langle X \rangle_W \subseteq \langle X \rangle$.

Theorem 2.5. Let V be a vector space over a field F and X be a finite subset of V. Then the following are equivalent:

1 – $X \subseteq \langle X \rangle_W$.

2 – $\langle X \rangle = \langle X \rangle_W$.

Proof. $(1) \Rightarrow (2)$. Suppose $X \subseteq \langle X \rangle_W$. Since $\langle X \rangle$ is the smallest subspace in V containing X, we have $\langle X \rangle \subseteq \langle X \rangle_W$. On the other hand, $\langle X \rangle_W \subseteq \langle X \rangle$ by Lemma 2.4. Thus, $\langle X \rangle = \langle X \rangle_W$.

$(2) \Rightarrow (1)$. Suppose $\langle X \rangle = \langle X \rangle_W$. Then by $X \subseteq \langle X \rangle$, implies that $X \subseteq \langle X \rangle_W$.

Corollary 2.6. Let V be a vector space over a field F, and X be a finite subset of V. If $V = \langle X \rangle_W$, then $V = \langle X \rangle$.

Notice. Theorem 2.5 shows that, $\langle X \rangle \neq \langle X \rangle_W$ if and only if $X \not\subset \langle X \rangle_W$, i.e., it is possible to generate a subspace weakly by sets that are not contained in them. This is shown in the following example:

Example. With R as the field of real numbers, let $X = \{(1,0),(0,-2)\}$ be a subset of the vector space R^2 over R. It is clear that $\langle X \rangle_W = \{(x, 2x);\, x \in R\}$, and that $X \not\subset \langle X \rangle_W$. Thus, by Theorem 2.5, implies that $\langle X \rangle \neq \langle X \rangle_W$.

Theorem 2.7. Let V be a vector space over a field F, and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. Then the following are equivalent:

1 – $X \not\subset \langle X \rangle_W$.

2 – There exists an element $v_i \in X$ where $1 \le i \le n$ such that for any elements $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ for which $v_i = \sum_{j=1}^{n} a_j v_j$ yields that $\sum_{j=1}^{n} a_j = 1$.

3 – $\langle X \rangle \ne \langle X \rangle_W$.

Proof. $(1) \Rightarrow (2)$. Suppose that $X \not\subset \langle X \rangle_W$, then there exists an element $v_i \in X$ where $1 \le i \le n$ such that $v_i \notin \langle X \rangle_W$, i.e., for any elements $a_1, a_2, \cdots, a_n \in F$ such that $v_i = \sum_{j=1}^{n} a_j v_j$, so $b = \sum_{j=1}^{n} a_j \ne 0$. We suppose that $b \ne 1$, i.e., $1 - b \ne 0$, then:

$v_i - bv_i = \sum_{j=1}^{n} a_j v_j - bv_i = \sum_{j=1}^{n} c_j v_j$

where $c_1, c_2, \cdots, c_n \in F$, with $c_i = a_i - b$, and $c_j = a_j$ for $i \ne j$. Therefore:

$v_i = \sum_{j=1}^{n} [(1-b)^{-1} c_j] v_j$

and

$\sum_{j=1}^{n} (1-b)^{-1} c_j = (1-b)^{-1} \sum_{j=1}^{n} c_j = (1-b)^{-1} [\sum_{j=1}^{n} a_j - b] = 0$

Which means that $v_i \in \langle X \rangle_W$, a contradiction. Therefore, $b = \sum_{j=1}^{n} a_j = 1$.

$(2) \Rightarrow (1)$. Obvious.

$(3) \Leftrightarrow (1)$. Direct by Theorem 2.5.

Example. With R as the field of real numbers, let $X = \{(1,0), (0,1), (2,-1)\}$ be a subset of the vector space R^2 over R.

We notice that $R^2 = \langle X \rangle$ and $\langle X \rangle_W = \{(x,-x) \,;\, x \in R\}$. The element $(2,-1)$ is written as the form**:**

$(2,-1) = \alpha(1,0) + \beta(0,1) + \gamma(2,-1)$.

where $\alpha = 2-2\gamma, \quad \beta = -1+\gamma ; \quad \gamma \in R$. It is easy to notice that $\alpha+\beta+\gamma=1$, $X \not\subset \langle X \rangle_W$ and $\langle X \rangle \neq \langle X \rangle_W$.

Lemma 2.8. Let V be a vector space over a field F, and X be a finite subset of V. If X is a linearly independent set, then $X \not\subset \langle X \rangle_W$, and $\langle X \rangle \neq \langle X \rangle_W$.

Proof. Direct by Theorem 2.7.

Theorem 2.9. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V where $0 \notin X$. Then the following are equivalent:

1 – The set X generates V over F, i.e., $V = \langle X \rangle$.

2 – The set $X \cup \{0\}$ weakly generates V over F, i.e., $V = \langle X \cup \{0\} \rangle_W$.

Proof. $(1) \Rightarrow (2)$. Suppose that X generates V over F, then for every $v \in V$ there exist elements $a_1, a_2, \cdots, a_n \in F$ such that $v = \sum_{i=1}^{n} a_i v_i$. Let $v_{n+1} = 0 \in V$, and $a_{n+1} = -\sum_{i=1}^{n} a_i \in F$, then $v = \sum_{i=1}^{n+1} a_i v_i$ and $\sum_{i=1}^{n+1} a_i = 0$. This indicates that $X \cup \{0\}$ weakly generates V over F, i.e., $V = \langle X \cup \{0\} \rangle_W$.

$(2) \Rightarrow (1)$. Suppose that the set $X \cup \{0\}$ weakly generates V over F, then for every $v \in V$ there exist elements $a_0, a_1, a_2, \cdots, a_n \in F$ such that $v = \sum_{i=0}^{n} a_i v_i$ and $\sum_{i=0}^{n} a_i = 0$ where $v_0 = 0$. Therefore, $v = \sum_{i=0}^{n} a_i v_i = \sum_{i=1}^{n} a_i v_i$, and this indicates that X generates V over F, i.e., $V = \langle X \rangle$.

Corollary 2.10. Let V be a vector space over a field F, and X be a finite subset of V. If $0 \in X$, then the following are equivalent:

1 – The set X generates V over F, i.e., $V = \langle X \rangle$.

2 – The set X weakly generates V over F, i.e., $V = \langle X \rangle_W$.

Lemma 2.11. Let V be a vector space over a field F, and let X , Y be finite subsets of V. The following hold:

1 – If $X \subseteq Y$, then $\langle X \rangle_W \subseteq \langle Y \rangle_W$.

2 – If $X \subseteq \langle Y \rangle_W$, then $\langle X \rangle_W \subseteq \langle Y \rangle_W$ and $\langle X \rangle \subseteq \langle Y \rangle_W$.

3 – If $X \subseteq \langle Y \rangle_W$ and $Y \subseteq \langle X \rangle_W$, then $\langle X \rangle = \langle Y \rangle$ and $\langle X \rangle_W = \langle Y \rangle_W$.

4 – $\langle X \rangle_W + \langle Y \rangle_W \subseteq \langle X + Y \rangle_W$.

Proof. 1, 2, and 3 are clear.

4 – For any $u \in \langle X \rangle_W + \langle Y \rangle_W$, there exists $x \in \langle X \rangle_W$ and $y \in \langle Y \rangle_W$ such that $u = x + y$. It is clear that $\langle X \rangle_W, \langle Y \rangle_W \subseteq \langle X \cup Y \rangle_W$ by (1), then $x, y \in \langle X \cup Y \rangle_W$. Thus, $u = x + y \in \langle X \cup Y \rangle_W$. Therefore, $\langle X \rangle_W + \langle Y \rangle_W \subseteq \langle X \cup Y \rangle_W$.

Notice. Let V be a vector space over a field F, and X , Y be finite subsets of V .Then the inclusion $\langle X \cup Y \rangle_W \subseteq \langle X \rangle_W + \langle Y \rangle_W$ is not valid in the general case. This is shown in the following example:

Example. With R as the field of real numbers, let $X = \{(2,3)\}$, and $Y = \{(1,0),(0,1)\}$ be subsets of the vector space R^2 over R. It is clear that:

$\langle X \rangle_W = \{(0,0)\}$

$\langle Y \rangle_W = \langle (1,-1) \rangle$

$\langle X \rangle_W + \langle Y \rangle_W = \langle (1,-1) \rangle$

and $\langle X \cup Y \rangle_W = R^2$. Therefore, $\langle X \cup Y \rangle_W \not\subset \langle X \rangle_W + \langle Y \rangle_W$.

3. Weak linear independence and full linear dependence.

In this section, we study a special type of finite subsets of a vector space which are considered a generalization of linearly independent sets. We start with the following definition:

Definition. Let V be a vector space over a field F and $X=\{v_1, v_2, \cdots, v_n\}$ be a subset of V. We say that X is *weakly linearly independent* or (*weakly independent* for short) if for any $a_1, a_2, \cdots, a_n \in F$ such that $\sum_{i=1}^{n} a_i v_i = 0$ and $\sum_{i=1}^{n} a_i = 0$ implies that $a_1 = a_2 = \cdots = a_n = 0$. If X is not weakly independent, then we say that X is *fully linearly dependent* or (*fully dependent* for short).

Example. With R as the field of real numbers, let $X=\{(1,0),(0,1),(2,3)\}$, and $Y=\{(1,0),(0,1),(2,-1)\}$ be subsets of the vector space R^2 over R. It is easy to show that X is weakly independent, while Y is fully dependent.

Lemma 3.1. Let V be a vector space over a field F. The following hold:

1 – Each subset of V consisting of two different elements is weakly independent.

2 – Each independent finite subset of V is weakly independent.

3 – Let $X=\{v_1, v_2, \cdots, v_n\}$ be an independent subset of V, then for any $v \in V$, the set $Y=\{v_1 - v, v_2 - v, \cdots, v_n - v\}$ is weakly independent.

Proof. 1 – Let $\{v_1, v_2\}$ be a subset of V, where $v_1 \neq v_2$ and $\alpha_1, \alpha_2 \in F$, such that:

$$\begin{cases} \alpha_1 v_1 + \alpha_2 v_2 = 0 \\ \alpha_1 + \alpha_2 = 0 \end{cases}$$

Then $\alpha_2 = -\alpha_1$ and $\alpha_1 v_1 - \alpha_1 v_2 = 0$, so $\alpha_1(v_1 - v_2) = 0$. We supposed that $\alpha_1 \neq 0$, then $v_1 = v_2$, a contradiction. Therefore, $\alpha_2 = \alpha_2 = 0$, i.e., $\{v_1, v_2\}$ is weakly independent.

2 – Let $X=\{v_1, v_2, \cdots, v_n\}$ be an independent finite subset of V, and let $a_1, a_2, \cdots, a_n \in F$ such that:

$\sum_{i=1}^{n} a_i v_i = 0$ and $\sum_{i=1}^{n} a_i = 0$.

Since X is independent, yields that $a_1 = a_2 = \cdots = a_n = 0$. Therefore, X is weakly independent.

3 – Let $a_1, a_2, \cdots, a_n \in F$ such that $\sum_{i=1}^{n} a_i (v_i - v) = 0$ and $\sum_{i=1}^{n} a_i = 0$. Then:

$\sum_{i=1}^{n} a_i (v_i - v) = \sum_{i=1}^{n} a_i v_i - (\sum_{i=1}^{n} a_i) v = \sum_{i=1}^{n} a_i v_i = 0$.

Since X is independent, yields that $a_1 = a_2 = \cdots = a_n = 0$. Thus, Y is weakly independent.

According to Lemma 3.1, we found that each independent set is weakly independent, but the opposite is not true in the general case, i.e., if X is weakly independent, then this does not necessarily mean that X is independent. This is shown in the following example:

Example. With R as the field of real numbers, the set $X = \{(1,0), (0,1), (2,3)\}$ of the vector space R^2 over R is weakly independent, but it is clear that X is not independent.

We state the relationship between full linear dependence and linear dependence in the following lemma:

Lemma 3.2. Let V be a vector space over a field F. The following hold:

1 – Each fully dependent finite subset of V is dependent.

2 – Each dependent finite subset of V is either weakly independent or fully dependent.

Proof. 1 – Let $X = \{v_1, v_2, \cdots, v_n\}$ be a fully dependent subset of V, then there are $a_1, a_2, \cdots, a_n \in F$ not all zero such that $\sum_{i=1}^{n} a_i = 0$ and $\sum_{i=1}^{n} a_i v_i = 0$. Thus X is dependent.

2 – Let $X = \{v_1, v_2, \cdots, v_n\}$ be a dependent subset of V, then X is not independent. Let $a_1, a_2, \cdots, a_n \in F$ such that $\sum_{i=1}^{n} a_i = 0$ and $\sum_{i=1}^{n} a_i v_i = 0$, then we recognize two cases:

i – If $a_1 = a_2 = \cdots = a_n = 0$, then X is weakly independent.

ii – If $a_1, a_2, \cdots, a_n$ not all zero, then X is fully dependent.

According to Lemma 3.2, we found that each fully dependent set is dependent, but the opposite is not true in the general case, i.e., if X is dependent, then this does not necessarily mean that X is fully dependent. This is shown in the following example:

Example. With R as the field of real numbers, let $X = \{(1,0),(0,1),(2,3)\}$ be a subset of R^2 over R. It is clear that X is dependent but is not fully dependent.

Let V be a vector space over a field F and X be a finite subset of V. It is known that if $0 \in X$, then X is dependent. The following lemma shows the necessary and sufficient condition for X to be independent.

Lemma 3.3. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V such that $0 \notin X$. Then the following are equivalent:

1 – X is independent.

2 – $X \cup \{0\}$ is weakly independent.

Proof. $(1) \Rightarrow (2)$. With $v_0 = 0$, let $\beta_0, \beta_1, \beta_2, \cdots, \beta_n \in F$ such that $\sum_{i=0}^{n} \beta_i v_i = 0$ and $\sum_{i=0}^{n} \beta_i = 0$. So, $\sum_{i=1}^{n} \beta_i v_i = 0$ and $\sum_{i=1}^{n} \beta_i = 0$. Since X is independent, then $\beta_1 = \beta_2 = \cdots = \beta_n = 0$, and since $\sum_{i=0}^{n} \beta_i v_i = 0$, we find that $\beta_0 = 0$. Therefore, $X \cup \{0\}$ is weakly independent.

$(2) \Rightarrow (1)$. Suppose $X \cup \{0\} = \{v_0, v_1, v_2, \cdots, v_n\}$ is weakly independent, where $v_0 = 0$. Let $\beta_1, \beta_2, \cdots, \beta_n \in F$ such that $\sum_{i=1}^{n} \beta_i v_i = 0$, then for $\beta_0 = -\sum_{i=1}^{n} \beta_i$ we find that:

$\sum_{i=0}^{n} \beta_i = 0$ and $\sum_{i=0}^{n} \beta_i v_i = 0$

Then by assumption $\beta_0 = \beta_1 = \beta_2 = \cdots = \beta_n = 0$. Thus, $\beta_1 = \beta_2 = \cdots = \beta_n = 0$. Therefore, X is independent.

According to Lemma 3.3, we can formulate the following corollary:

Corollary 3.4. Let V be a vector space over a field F and X be a finite subset of V such that $0 \notin X$. Then the following are equivalent:

1 – X is dependent.

2 – $X \cup \{0\}$ is fully dependent.

Lemma 3.5. Let V be a vector space over a field F. The following hold:

1 – For each non-zero element $v \in V$, then $\{v, 0\}$ is weakly independent.

2 – The set $\{0\}$ is weakly independent.

Proof. 1 – Let $v \in V$ be a non-zero element. Since $\{v\}$ is independent, then $\{v, 0\}$ is weakly independent by Lemma 3.3.

2 – Obvious.

Lemma 3.6. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. The following hold:

1 – If X is weakly independent, then every non-empty subset of X is weakly independent.

2 – If there exists a non-empty fully dependent subset of X, then X is fully dependent.

Proof. 1 – Without loss of generality, suppose that $Y = \{v_1, v_2, \cdots, v_m\}$, where $m \le n$ is a subset of X, and $a_1, a_2, \cdots, a_m \in F$ such that:

$\sum_{i=1}^{m} a_i v_i = 0$ and $\sum_{i=1}^{m} a_i = 0$

then

$\sum_{i=1}^{n} a_i v_i = 0$ and $\sum_{i=1}^{n} a_i = 0$

where $a_i = 0$ for $m < i \le n$. Since X is weakly independent, then:

$a_1 = a_2 = \cdots = a_n = 0$

Therefore, $a_1 = a_2 = \cdots = a_m = 0$, i.e., X is weakly independent.

2 – Without loss of generality, suppose that $Y=\{v_1, v_2, \cdots, v_m\}$, where $m \le n$ is a fully dependent subset of X, then there exist elements $a_1, a_2, \cdots, a_m \in F$ not all zero such that:

$\sum_{i=1}^{m} a_i v_i = 0$ and $\sum_{i=1}^{m} a_i = 0$.

Let $a_1, a_2, \cdots, a_n \in F$ such that $a_i = 0$ for $m < i \le n$, then $\sum_{i=1}^{n} a_i v_i = 0$, $\sum_{i=1}^{n} a_i = 0$, and the elements $a_1, a_2, \cdots, a_n \in F$ not all zero. Therefore, X is fully dependent.

Two equivalent conditions for weak independence are presented by the following theorem:

Theorem 3.7. Let V be a vector space over a field F and $X=\{v_1, v_2, \cdots, v_n\}$ be a subset of V. The following are equivalent:

1 – X is weakly independent.

2 – The zero element of V is written as a weak linear combination of X as the only form.

3 – Each $v \in \langle X \rangle_W$ is written as a weak linear combination of X as the only form.

Proof. $(1) \Rightarrow (2)$. Let $a_1, a_2, \cdots, a_n \in F$ such that $\sum_{i=1}^{n} a_i v_i = 0$ and $\sum_{i=1}^{n} a_i = 0$.

Since X is weakly independent, then $a_1 = a_2 = \cdots = a_n = 0$. Thus, the zero element of V is written as a weak linear combination of X as the only form.

$(2) \Rightarrow (3)$. Let $v \in \langle X \rangle_W$, then there exist $a_1, a_2, \cdots, a_n \in F$ such that:

$v = \sum_{i=1}^{n} a_i v_i$ and $\sum_{i=1}^{n} a_i = 0$.

Let $\beta_1, \beta_2, \cdots, \beta_n \in F$ such that $v = \sum_{i=1}^{n} \beta_i v_i$ and $\sum_{i=1}^{n} \beta_i = 0$.

Then,

$\sum_{i=1}^{n} (\alpha_i - \beta_i) v_i = \sum_{i=1}^{n} \alpha_i v_i - \sum_{i=1}^{n} \beta_i v_i = 0$

$\sum_{i=1}^{n}(\alpha_i - \beta_i) = \sum_{i=1}^{n}\alpha_i - \sum_{i=1}^{n}\beta_i = 0$

Then, by assumption, $\alpha_i - \beta_i = 0$ for each $1 \le i \le n$. Then $\alpha_i = \beta_i$ for each $1 \le i \le n$.Thus, v is written as a weak linear combination of X as the only form.

$(3) \Rightarrow (1)$. Let $a_1, a_2, \cdots, a_n \in F$, such that $\sum_{i=1}^{n} a_i v_i = 0$ and $\sum_{i=1}^{n} a_i = 0$. Since $0 \in \langle X \rangle_W$, then by assumption, $a_1 = a_2 = \cdots = a_n = 0$. Thus, X is weakly independent.

Theorem 3.8. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. The following hold:

1 – If X is weakly independent and non-independent, then $X \subset \langle X \rangle_W$ and $\langle X \rangle = \langle X \rangle_W$.

2 – If X is dependent such that $X \not\subset \langle X \rangle_W$, then X is fully dependent.

Proof. 1 – Suppose that X is weakly independent and non-independent, then we recognize the following cases:

– $0 \in X$. Suppose that $v_1 = 0$, then for each $v_i \in X$ where $2 \le i \le n$:

$v_i = -1v_1 + 0v_2 + \cdots + 0v_{i-1} + 1v_i + 0v_{i+1} + \cdots + 0v_n$

Thus, $v_i \in \langle X \rangle_W$ where $2 \le i \le n$. Since $v_1 = 0 \in \langle X \rangle_W$, then $X \subseteq \langle X \rangle_W$.

– $0 \notin X$. Since X is non-independent, then there exist $a_1, a_2, \cdots, a_n$ not all zero in F such that $\sum_{i=1}^{n} a_i v_i = 0$. It is clear that $\beta = \sum_{i=1}^{n} a_i \ne 0$ because if $\beta = \sum_{i=1}^{n} a_i = 0$, then since X is weakly independent, yields that:

$a_1 = a_2 = \cdots = a_n = 0$

and this is a contradiction. Thus, $\beta = \sum_{i=1}^{n} a_i \ne 0$, then:

$\sum_{j=1}^{n}(-\beta)^{-1}a_j v_j = 0$ and $\sum_{j=1}^{n}(-\beta)^{-1}a_j = -1$

Then, for each $v_i \in X$ where $1 \le i \le n$ we find that:

$v_i = v_i + \sum_{j=1}^{n}(-\beta)^{-1}a_j v_j$ and $1 + \sum_{j=1}^{n}(-\beta)^{-1}a_j = 0$

Thus, $v_i \in \langle X \rangle_W$, i.e., $X \subseteq \langle X \rangle_W$. Therefore, $\langle X \rangle = \langle X \rangle_W$ by Lemma 2.4.

2 – Direct by (1).

Theorem 3.9. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a weakly independent and non-independent subset of V. If there exists an element $v \in X$ can be written as a linear combination of $X \setminus \{v\}$, then $X \setminus \{v\}$ is independent.

Proof. Suppose that X is weakly independent and non-independent, then $X \subset \langle X \rangle_W$, and $\langle X \rangle = \langle X \rangle_W$ by Theorem 3.8. Moreover, there exists an element, let it be $v_1 \in X$, can be written as a linear combination of $Y = \{v_2, \cdots, v_n\}$, then $v_1 \in \langle Y \rangle$, and $\langle X \rangle = \langle Y \rangle$. On the other hand, by Lemma 3.6, Y is weakly independent. We suppose that Y is dependent, then $Y \subset \langle Y \rangle_W$ and $\langle Y \rangle = \langle Y \rangle_W$ by Theorem 3.8. This indicates that $\langle Y \rangle_W = \langle X \rangle_W$. Let $u = v_1 - v_2 \in \langle X \rangle_W$, then $u \in \langle Y \rangle_W$, i.e., there exist $a_2, \cdots, a_n \in F$ such that

$u = \sum_{i=2}^{n} \alpha_i v_i$ and $\sum_{i=2}^{n} \alpha_i = 0$

Hence, the element $u \in \langle X \rangle_W$ can be written as a weak linear combination of X as two different forms and this is contradictory to Theorem 3.7. Therefore, Y is independent, i.e., $X \setminus \{v_1\}$ is independent.

Notice. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a weakly independent and non-independent subset of V. If $v \in X$, then $X \setminus \{v\}$ is not independent in the general case. This is shown in the following example:

Example. With R as the field of real numbers, let $X = \{(1,0),(0,1),(2,0)\}$, be a subset of the vector space R^2 over R. It is easy to show that X is weakly independent and $Y = \{(1,0),(2,0)\}$ is dependent.

Now, we state the basic properties of the fully dependent set. We start with the following theorem:

Theorem 3.10. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. The following are equivalent:

1 – X is fully dependent.

2 – There exists an element $v_j \in X$; $(1 \le j \le n)$, for which there are

$a_1, a_2, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n \in F$

such that $v_j = \sum_{i=1, i\neq j}^{n} a_i v_i$ and $\sum_{i=1, i\neq j}^{n} a_i = 1$.

Proof. $(1) \Rightarrow (2)$. Suppose that X is fully linearly dependent, then there exist $a_1, a_2, \cdots, a_n \in F$ not all zero such that:

$\sum_{i=1}^{n} a_i v_i = 0$ and $\sum_{i=1}^{n} a_i v_i = 0$

Let $a_j \neq 0;\ (1 \le j \le n)$, then $a_j = -\sum_{i=1, i\neq j}^{n} a_i$ and $a_j v_j = -\sum_{i=1, i\neq j}^{n} a_i v_i$. Hence,

$v_j = 1.v_j = (a_j^{-1} a_j) v_j = a_j^{-1}(a_j v_j) = \sum_{i=1,\, i\neq j}^{n} (-a_j^{-1} a_i) v_i$

and

$\sum_{i=1,\, i\neq j}^{n} (-a_j^{-1} a_i) = a_j^{-1}(-\sum_{i=1,\, i\neq j}^{n} a_i) = a_j^{-1} a_j = 1$

$(2) \Rightarrow (1)$. Let $v_j \in X$; $(1 \le j \le n)$, for which there are:

$a_1, a_2, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n \in F$

such that $v_j = \sum_{i=1, i\neq j}^{n} a_i v_i$ and $\sum_{i=1, i\neq j}^{n} a_i = 1$, then for $a_j = -1$ yields that:

$\sum_{i=1}^{n} a_i v_i = 0$ and $\sum_{i=1}^{n} a_i v_i = 0$

and $a_1, a_2, \cdots, a_n \in F$ not all zero. Therefore, X is fully dependent.

According to the last theorem, we can form the following theorem:

Theorem 3.11. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. Then the following are equivalent:

1 – X is weakly independent and non-independent.

2 – If $v_j = \sum_{i=1, i\neq j}^{n} a_i v_i$; $(1 \leq j \leq n)$ where $a_1, a_2, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n \in F$, then $\sum_{i=1, i\neq j}^{n} a_i \neq 1$.

Proof. Direct by Theorem 3.10.

According to Theorem 3.9 and Theorem 3.11, we can form the following corollary:

Corollary 3.12. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be an independent subset of V. If $v \in \langle X \rangle$ such that $v = \sum_{i=1}^{n} a_i v_i$ and $\sum_{i=1}^{n} a_i v_i \neq 1$ where $a_1, a_2, \cdots, a_n \in F$, then $X \cup \{v\}$ is weakly independent.

Theorem 3.13. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a fully dependent subset of V. If X weakly generates V, then there exists an element $v_j \in X$; $(1 \leq j \leq n)$ such that $X \setminus \{v_j\}$ weakly generates V.

Proof. Suppose that X is fully dependent, and $V = \langle X \rangle_W$, then for each $v \in V$ there exist $a_1, a_2, \cdots, a_n \in F$ such that $v = \sum_{i=1}^{n} a_i v_i$ and $\sum_{i=1}^{n} a_i = 0$.

By Theorem 3.10 there exists an element $v_j \in X\ ;\ (1 \le j \le n)$, for which there are

$\beta_1, \beta_2, \cdots, \beta_{j-1}, \beta_{j+1}, \cdots, \beta_n \in F$

such that, $v_j = \sum_{i=1, i\neq j}^{n} \beta_i v_i$ and $\sum_{i=1, i\neq j}^{n} \beta_i = 1$. Hence, $v = \sum_{i=1,\ i\neq j}^{n} (a_i + a_j \beta_i) v_i$.

Since $\sum_{i=1, i\neq j}^{n} \beta_i = 1$, then

$\sum_{i=1,\ i\neq j}^{n} (a_i + a_j \beta_i) = \sum_{i=1,\ i\neq j}^{n} a_i + a_j \sum_{i=1,\ i\neq j}^{n} \beta_i = \sum_{i=1}^{n} a_i = 0$

Therefore, $X \setminus \{v_j\}$ weakly generates V.

We state a special type of weakly independent sets and its properties, we start with the following definition:

Definition. Let V be a vector space over a field F and X be a weakly independent finite subset of V. We say that X is *maximal weakly linearly independent* or (*maximal weakly independent* for short) if for all $v \in V$ where $v \notin X$ implies that $X \cup \{v\}$ is fully dependent.

Lemma 3.14. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a maximal weakly independent subset of V. Then, X is dependent.

Proof. Suppose that X is maximal weakly independent. We suppose that X is independent, then $0 \notin X$. Thus, $X \cup \{0\}$ is weakly independent by Lemma 3.3, a contradiction. Therefore, X is dependent.

Let V be a vector space over a field F and X be a finite subset of V. It is known that if X is maximal independent, then for all $v \in V$ where $v \notin X$, implies that $X \cup \{v\}$ is dependent. The following lemma shows the necessary and sufficient condition for X to be maximal independent.

Lemma 3.15. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V, such that $0 \notin X$. Then the following are equivalent:

1 – X is maximal independent.
2 – $X \cup \{0\}$ is maximal weakly independent.

Proof. $(1) \Rightarrow (2)$. Suppose that X is maximal independent, so X is independent. Thus, $X \cup \{0\}$ is weakly independent by Lemma 3.3. With $v_0 = 0$ and $v_{n+1} \in V$, such that $v_{n+1} \notin X$, let $\beta_0, \beta_1, \beta_2, \cdots, \beta_{n+1} \in F$, such that

$\sum_{i=0}^{n+1} \beta_i v_i = 0$ and $\sum_{i=0}^{n+1} \beta_i = 0$

Then,

$\sum_{i=1}^{n+1} \beta_i v_i = \sum_{i=0}^{n+1} \beta_i v_i = 0$

Since X is maximal independent, then $\beta_1, \beta_2, \cdots, \beta_{n+1} \in F$ not all zero, i.e., $\beta_0, \beta_1, \cdots, \beta_{n+1} \in F$ not all zero where $\beta_0 = -\sum_{i=1}^{n+1} \beta_i$. Therefore, $X \cup \{0\}$ is maximal weakly independent.

$(2) \Rightarrow (1)$. Suppose that $X \cup \{0\}$ is maximal weakly independent, so $X \cup \{0\}$ is weakly independent. Thus, X is independent by Lemma 3.3. With $v_0 = 0$ and $v_{n+1} \in V$, such that $v_{n+1} \notin X$, let $\beta_1, \beta_2, \cdots, \beta_{n+1} \in F$ such that $\sum_{i=1}^{n+1} \beta_i v_i = 0$.

We suppose that $\beta_0 = -\sum_{i=1}^{n+1} \beta_i$, then:

$\sum_{i=1}^{n+1} \beta_i v_i = \sum_{i=0}^{n+1} \beta_i v_i = 0$ and $\sum_{i=0}^{n+1} \beta_i = 0$

Since $X \cup \{0\}$ is maximal weakly independent, then $\beta_0, \beta_1, \cdots, \beta_{n+1} \in F$ not all zero, i.e., $\beta_1, \beta_2, \cdots, \beta_{n+1} \in F$ not all zero. Therefore, X is maximal independent.

Theorem 3.16. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a maximal weakly independent subset of V. Then $V = \langle X \rangle_W$.

Proof. Suppose that X is maximal weakly independent, then we recognize the following cases:

– $0 \in X$. Then $X \setminus \{0\}$ is maximal independent by Lemma 3.15. Then, $X \setminus \{0\}$ generates V . Hence, X weakly generates V by Theorem 2.8.
– $0 \notin X$. Then X is dependent by Lemma 3.14. Then, there exists an element, let it be $v_1 \in X$, can be written as a linear combination of $Y = \{v_2, \cdots, v_n\}$, i.e., there exist $\beta_2, \cdots, \beta_n \in F$ such that $v_1 = \sum_{i=2}^{n} \beta_i v_i$. Moreover, Y is independent by Theorem 3.9. We suppose that Y is not maximal, then there exists an element $v_0 \in V$ where $v_0 \notin X$ such that $Y_1 = \{v_0, v_2, \cdots, v_n\}$ is independent. On the other hand, since X is maximal weakly independent, then $X \cup \{v_0\}$ is fully dependent, so there exist $a_0, a_1, \cdots, a_n \in F$ not all zero such that $\sum_{i=0}^{n} a_i = 0$ and $\sum_{i=0}^{n} a_i v_i = 0$. It is clear that $a_0 \neq 0$, because if $a_0 = 0$ then:
$\sum_{i=1}^{n} a_i = 0$ and $\sum_{i=1}^{n} a_i v_i = 0$
Since X is weakly independent, yields that $a_0 = a_1 = \cdots = a_n = 0$, and this is contradictory to $X \cup \{v_0\}$ is fully dependent. Thus, v_0 can be written as a linear combination of X , i.e., there exist $\gamma_1, \cdots, \gamma_n \in F$ such that $v_0 = \sum_{i=1}^{n} \gamma_i v_i$, then
$$v_0 = \gamma_1 v_1 + \sum_{i=2}^{n} \gamma_i v_i = \sum_{i=2}^{n} (\gamma_1 \beta_i + \gamma_i) v_i$$
Thus, v_0 can be written as a linear combination of Y , i.e., Y_1 is dependent, a contradiction. Hence, Y is maximal independent, then $V = \langle Y \rangle = \langle X \rangle$. Since X is maximal weakly independent, then X is dependent by Lemma 3.14. Also, $\langle X \rangle_W = \langle X \rangle$ by Theorem 3.8. Therefore, $V = \langle X \rangle_W$.

According to the last theorem, we can form the following corollary:

Corollary 3.17. Let V be a vector space over a field F and $X=\{v_1, v_2, \cdots, v_n\}$ be a maximal weakly independent subset of V. Then, there exists an element $v \in X$ can be written as a linear combination of $X \setminus \{v\}$, such that $X \setminus \{v\}$ is maximal independent.

Theorem 3.18. Let V be a vector space over a field F, and X, Y are finite subsets of V. The following hold:

1 – If X and Y are maximal weakly independent, then $Card\ X = Card\ Y$.

2 – If X is maximal independent, and Y is maximal weakly independent, then $Card\ Y = Card\ X + 1$.

Proof. 1 – Suppose that X and Y are maximal weakly independent, then by Corollary 3.17, there exists an element $v \in X$ such that $X \setminus \{v\}$ is maximal independent. Also, there exists an element $u \in Y$ such that $Y \setminus \{u\}$ is maximal independent. Thus, $Card\ X \setminus \{v\} = Card\ Y \setminus \{u\}$. Therefore, $Card\ X = Card\ Y$.

2 – Suppose that X is maximal independent, and Y is maximal weakly independent, then by Corollary 3.17, there exists an element $v \in Y$ such that $Y \setminus \{v\}$ is maximal independent, Thus, $Card\ X = Card\ Y \setminus \{v\}$. Therefore, $Card\ Y = Card\ X + 1$.

Theorem 3.19. Let V be a vector space over a field F and $X=\{v_1, v_2, \cdots, v_n\}$ be a subset of V. Then for any element $v_0 \in \langle X \rangle$ such that $v_0 = \sum_{i=1}^{n} a_i v_i$ and $\sum_{i=1}^{n} a_i \neq 1$ where $a_1, a_2, \cdots, a_n \in F$, the following hold:

1 – If X is independent, then $X \cup \{v_0\}$ is weakly independent.

2 – If X is maximal independent, then $X \cup \{v_0\}$ is maximal weakly independent.

Proof. 1 – Suppose that X is independent and $v_0 \in \langle X \rangle$ such that $v_0 = \sum_{i=1}^{n} a_i v_i$ and $\sum_{i=1}^{n} a_i \neq 1$ where $a_1, a_2, \cdots, a_n \in F$. We

supposed that $X \cup \{v_0\}$ is not weakly independent, i.e., $X \cup \{v_0\}$ is fully dependent, then there exist elements $\beta_0, \beta_1, \beta_2, \cdots, \beta_n \in F$ not all zero such that $\sum_{i=0}^{n} \beta_i v_i = 0$ and $\sum_{i=0}^{n} \beta_i = 0$. It is clear that $\beta_0 \neq 0$ because if $\beta_0 = 0$, then $\sum_{i=1}^{n} \beta_i v_i = 0$ and $\sum_{i=1}^{n} \beta_i = 0$, and since X is independent, yields that $\beta_0 = \beta_1 = \cdots = \beta_n = 0$, and this is contradictory to $\beta_0, \beta_1, \beta_2, \cdots, \beta_n \in F$ not all zero. So $\beta_0 \neq 0$, then $-\beta_0 = \sum_{i=1}^{n} \beta_i$ and $\sum_{i=1}^{n} (-\beta_0)^{-1} \beta_i = 1$. Therefore,
$\sum_{i=0}^{n} \beta_i v_i = \beta_0 v_0 + \sum_{i=1}^{n} \beta_i v_i = \sum_{i=1}^{n} (\beta_0 a_i + \beta_i) v_i = 0$
Since X is independent, yields that $\beta_0 a_i + \beta_i = 0$ for each $1 \leq i \leq n$. Hence $\sum_{i=1}^{n} (\beta_0 a_i + \beta_i) = 0$, then $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} (\beta_0)^{-1} \beta_i = 1$, a contradiction. Therefore, $X \cup \{v_0\}$ is weakly independent.

2 – Suppose that X is maximal independent, then $V = \langle X \rangle$, and X is independent. Let $v_0 \in V$ such that $v_0 = \sum_{i=1}^{n} a_i v_i$ and $\sum_{i=1}^{n} a_i \neq 1$ where $a_1, a_2, \cdots, a_n \in F$. Then, $X \cup \{v_0\}$ is weakly independent by (1). We supposed that $X \cup \{v_0\}$ is not maximal weakly independent, then there exists an element $v_{n+1} \in V$ such that $X \cup \{v_0, v_{n+1}\}$ is weakly independent. Since v_0 can be written as a linear combination of X, then $v_0 = \sum_{i=1}^{n+1} a_i v_i$ where $a_{n+1} = 0$. Thus, v_0 can be written as a linear combination of $X \cup \{v_{n+1}\}$, then $X \cup \{v_{n+1}\}$ is independent by Theorem 3.9, a contradiction. Therefore, $X \cup \{v_0\}$ is maximal weakly independent.

Lemma 3.20. Let V be a vector space over a field F and X be a finite subset of V such that $V = \langle X \rangle_W$. Then, there exists a subset X' of X such that $V = \langle X' \rangle$, and X' is maximal independent.

Proof. Obvious.
We state a special type of weakly generated sets and its properties. We start with the following definition:
Definition. Let V be a vector space over a field F and X be a finite subset of V. We say that X is a *minimal weakly generated set* of V if it satisfies the following:
1 – $V = \langle X \rangle_W$.
2 – No proper subset of X weakly generates V.
More precisely, $V = \langle X \rangle_W$ and $V \neq \langle X \setminus \{v\} \rangle_W$ for all $v \in X$.
Theorem 3.21. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. The following hold:
1 – If X is a minimal weakly generated set of V, then X is weakly independent.
2 – If X is maximal weakly independent, then X is a minimal weakly generated set of V.
Proof. 1 – Suppose that X is a minimal weakly generated set of V. We suppose that X is fully dependent, then by Theorem 3.13, there exists an element $v_j \in X$; $(1 \leq j \leq n)$ such that $X \setminus \{v_j\}$ weakly generates V, a contradiction. Therefore, X is weakly independent.
2 – We Suppose that $Y = \{v_1, v_2, \cdots, v_r\}$ is a subset of X such that $V = \langle Y \rangle_W$ and $X \neq Y$, then Y is weakly independent by Lemma 3.6. We recognize two cases:
– Y is independent. Then, by assumption and Lemma 2.8, $Y \not\subset \langle Y \rangle_W = V$, a contradiction. Then $X = Y$.
– Y is not independent. Then, by assumption and Theorem 3.8, $Y \subset \langle Y \rangle_W = V$. By Lemma 3.20, there exists a subset Y' of Y such that $V = \langle Y' \rangle$, and Y' is maximal independent, then $Card\, Y' = n - 1$ by Theorem 3.18. We recognize the following cases:
– $Card\, Y' = r - 1$, then $n = r$, since, $Y \subset X$, then $X = Y$.

– $Card\, Y' < r-1$, then $n < r$, a contradiction. Then, $X = Y$.
– $Card\, Y' > r-1$, then $Card\, Y' = r$, i.e., $Card\, Y' = Card\, Y$.
Since $Y' \subset Y$, then $Y' = Y$, a contradiction, where Y' is independent and Y is not. Then, $X = Y$.
Therefore, X is a minimal weakly generated set of V.

Corollary 3.22. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. By Theorem 3.7, if X is maximal weakly linearly independent, then every element $v \in \langle X \rangle_W$ can be written as a weak linear combination of X as the only form and by Theorem 3.16, $V = \langle X \rangle_W$. So, if X is maximal weakly linearly independent, then every element $v \in V$ can be written as a weak linear combination of X as the only form and in this case, X is a minimal weakly generated subset of V by Theorem 3.21.

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