The Best Uniform Polynomial Approximation To Class Of The Form $\frac{1}{a^4-x^4}$

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Abstract

It is very useful to be able to replace any given function by a simpler function, such as a polynomial, chosen to have values not identical with but very close to those of the given function.
And as it is known best approximation by polynomials is an important subject in approximation theory and has a large number of applications such as the numerical computation of a best approximating polynomial , partial differential equations , differential equations etc…..
In this paper we determine the best uniform polynomial approximation out of P_{4n} (the space of polynomials of degree at most $4n$)to a class of rational functions of the form $\frac{1}{a^4-x^4}$ on $[-c, c]$.
In this way we introduce a theorem about the best approximation of this class of rational function and we also obtain Chebyshev alternative set of these classes of functions.

Keywords: Uniform approximation, Chebyshev alternative, Chebyshev polynomials.

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حدودية التقريب المنتظم الأمثل لصفوف الدوال $\frac{1}{a^4-x^4}$

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الملخص

تُعد مسألة استبدال دالة ما بدالة أخرى أبسط منها تركيباً، مثل الحدوديات من المسائل المهمة والمفيدة جداً .

وكما نعلم فإنّ التقريب الأمثل بواسطة الحدوديات من المواضيع المهمة التي تعالجها نظرية التقريبات، ولها العديد من التطبيقات في مختلف المجالات، مثل الحسابات العددية ،والمعادلات التفاضلية، والمعادلات التفاضلية الجزئية إلخ...

نقدم في هذه الورقة البحثية حدودية التقريب المنتظم الأمثل لصفوف التوابع الكسرية من الشكل $\frac{1}{a^4-x^4}$ في الفضاء P_{4n} (فضاء كثيرات الحدود الجبرية من الدرجة $4n$) على المجال $[-c, c]$ حيث $a^2 > c^2$.

وبهذا الخصوص أثبتنا نظرية حول التقريب المنتظم لصفوف هذه الدوال ،وحدّدنا مجموعة متناوبات تشيبتشيف لصفوف هذه الدوال.

الكلمات المفتاحية: التقريب المنتظم – متناوبات تشيبتشيف – كثيرات حدود تشيبتشيف.

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1. Introduction.

Let $[\mathrm{b}, \mathrm{d}]$ be a closed and bounded interval of the real line . A space of continuous real valued functions on $[b,d]$ is denoted C[b,d] , That's:

$C[b,d] = \{f: [b,d] \to \mathbb{R}: f \text{ is continuous}\}$.

For all functions $f \in C[b,d]$, the uniform norm or $L_{\infty\_}$ - norm is defined by

$\|f\|_\infty = max_{x \in [b,d]} |f(x)|$.

Definition 1. [2]

Let $f \in C[b,d]$,there exists a unique polynomial $p_n^* \in P_n$ such that:

$\|f - p_n^*\|_\infty \le \|f - p_n\|_\infty \ , \ \forall p_n \in P_n$.

and p_n^* is called the best uniform polynomial approximation to f on $[b,d]$.

Where P_n is the space of algebraic polynomials of degree at most n.

The problem of existence and unique of such polynomial is studied in [2].

Theorem 1. [2] (Chebyshev Alternative Theorem)

Let $f \in C[b,d]$ and $\Delta(x) = f(x) - p(x)$, then $p(x)$ is the best uniform polynomial approximation to f on $[b,d]$ if and only if there exist at least $n+2$ points $x_1 < x_2 < \cdots < x_{n+2}$ in $[b,d]$ for whish

$$|\Delta(x_i)| = \max_{b \le x \le d} |f(x) - p(x)|$$

and at these points:

$\Delta(x_{i+1}) = -\Delta(x_i)\ ; i = 1,2,..,n+1$.

Definition 2 [3]. The Chebyshev polynomial in $[-1,1]$ is denoted by $T_n(x)$, where n is the degree of this polynomial .and is defined by:

$T_n(x) = \cos n\theta \ ; x = \cos\theta$.

And it has the recurrence formula:

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \ ; \ n = 2,3,..$$

where $T_0(x) = 1 \ , \ \ T_1(x) = x$.

Definition 2[3]. Shifted Chebyshev polynomial on **the interval** $[-c,c]$ is:

$$T_n^*(x) = \cos n\theta \quad , \cos\theta = \frac{x}{c}$$

Lemma 1. We have for $T_n(x) = \cos n\theta \quad , x = \cos\theta$, $|t| < 1$and k (natural number).

$\sum_{j=k}^{\infty} t^{4j} T_{4j}(x) = t^{4k} \frac{\cos 4k\theta - t^4 \sin 4\theta \sin 4k\theta - t^4 \cos 4\theta \cos 4k\theta}{1+t^8-2t^4\cos 4\theta}$.

Proof. In [4] it has been proved that:

$$\sum_{j=k}^{\infty} r^j T_{hj+c}(x) = r^k \frac{\cos(hk+c)\theta - r\sin h\theta \sin(hk+c)\theta - t^2 \cos h\theta \cos(hk+c)\theta}{1+r^2-2r\cos h\theta}$$

By substituting $r = t^4$, $h = 4$

and $c = 0$ we find what is required .

2. The Best Approximation of $\frac{1}{a^4-x^4}$.

Mehdi Dehghan and M.R.Eslachi found in[1], the best uniform polynomial approximation to $\frac{1}{a^2 \mp x^2}$ on $[-c, c]$ where $a^2 > c^2$ and they found that:

$$\frac{1}{a^2-x^2} = \frac{4t^2}{c^2(t^4-1)} - \frac{8t^2}{c^2(t^4-1)} \sum_{k=0}^{\infty} t^{2k} T_{2k}^*(x) \qquad (1)$$

$$\frac{1}{a^2+x^2} = \frac{4t^2}{c^2(t^4-1)} - \frac{8t^2}{c^2(t^4-1)} \sum_{k=0}^{\infty} (-1)^k t^{2k} T_{2k}^*(x) \qquad (2)$$

where t satisfies $t = \frac{a-\sqrt{a^2-c^2}}{c}$.

We used their expansion to expand our functions ,then we used the phase angle method to prove our theorem.

Theorem 2. The best uniform polynomial approximation out ofP_{4n} to $\frac{1}{a^4-x^4}$ on $[-c, c]$ where $a^2 > c^2$is:

$$p^*(x) \coloneqq p_{4n}(x) =$$

$$= \frac{4t^2}{a^2c^2(t^4-1)} - \frac{8t^2}{a^2c^2(t^4-1)} \sum_{k=0}^{n-1} t^{4k} T_{4k}^*(x) + \frac{4t^{4n+2}}{a^2c^2(t^4-1)(t^8-1)} T_{4n}^*(x) \qquad (3)$$

And $E_{4n}\left[\frac{\mathbf{1}}{\boldsymbol{a^4-x^4}},[-c,c]\right]=\frac{8t^{4n+6}}{a^2c^2(t^4-1)(t^8-1)}$.

where $t=\frac{a-\sqrt{a^2-c^2}}{c}$.

Proof. Let's first expand our function by Chebyshev series.
Using (1) and (2) we can find:

$$\frac{\mathbf{1}}{\boldsymbol{a^4-x^4}}=\frac{1}{2\boldsymbol{a^2}}\left(\frac{\mathbf{1}}{\boldsymbol{a^2+x^2}}+\frac{\mathbf{1}}{\boldsymbol{a^2-x^2}}\right)=$$

$$\frac{1}{2\boldsymbol{a^2}}\left[\left(\frac{4t^2}{c^2(t^4-1)}-\frac{8t^2}{c^2(t^4-1)}\sum_{k=0}^{\infty}(-1)^k t^{2k}T_{2k}^*(x)\right)-\frac{4t^2}{c^2(t^4-1)}-\frac{8t^2}{c^2(t^4-1)}\sum_{k=0}^{\infty}t^{2k}T_{2k}^*(x)\right]$$

$$=\frac{-4t^2}{a^2c^2(t^4-1)}-\frac{8t^2}{a^2c^2(t^4-1)}\sum_{k=0}^{\infty}t^{4k}T_{4k}^*(x) \qquad (4)$$

To prove that $p^*$is the best uniform polynomial approximation out of $[-c,c]$ to $\frac{\mathbf{1}}{\boldsymbol{a^4-x^4}}$, we must show that the difference

$$\Delta(x)=\frac{\mathbf{1}}{\boldsymbol{a^4-x^4}}-p^*(x) \qquad (5)$$

has $(4n+2)$ alternative points in $[-c,c]$. From (3) and (4) we have:

$$\Delta(x)=-\frac{8t^2}{a^2c^2(t^4-1)}\sum_{j=n}^{\infty}t^{4k}T_{4n}^*(x)-\frac{8t^{4n+2}}{a^2c^2(t^4-1)(t^8-1)}T_{4n}^*(x)$$

According to lemma 1 we obtain:

$$\Delta(x)=-\frac{8t^2}{a^2c^2(t^4-1)}\left[t^{4n}\frac{\cos 4n\theta-t^4\sin 4\theta\sin 4n\theta-t^4\cos 4\theta\cos 4n\theta}{1+t^8-2t^4\cos 4\theta}\right]-\frac{8t^{4n+2}}{a^2c^2(t^4-1)(t^8-1)}\cos 4n\theta$$

$$=-\frac{8t^{4n+2}}{a^2c^2(t^4-1)}\left[\frac{\cos 4n\theta-t^4\sin 4\theta\sin 4n\theta-t^4\cos 4\theta\cos 4n\theta}{1+t^8-2t^4\cos 4\theta}+\frac{1}{(t^8-1)}\cos 4n\theta\right]$$

Simplifying the previous formula using the following relations:

$$1+t^8-2t^4\cos 4\theta=4t^4\left[\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2\right]$$

$$T_2^*(x)=2\frac{x^2}{c^2}-1$$

$$t^8-1=4t^4\left(2\frac{a^2}{c^2}-1\right)^2-2(t^4+1)\ .$$

$$t^8-1=-4t^4\sqrt{\left(2\frac{a^2}{c^2}-1\right)^4-\left(2\frac{a^2}{c^2}-1\right)^2}$$

We find:

$$\Delta(x)=-\frac{8t^{4n+2}}{a^2c^2(t^4-1)}\Bigg[\frac{(t^8-1)(1-t^4\cos4\theta)+(1+t^8-2t^4\cos4\theta)}{1+t^8-2t^4\cos4\theta}\cos4n\theta-\frac{t^4(t^8-1)\sin4\theta}{(t^8-1)(1+t^8-2t^4\cos4\theta)}\sin4n\theta\Bigg]$$

Replacing $\cos4\theta=2(\cos2\theta)^2-1=2\left(T_2^*(x)\right)^2-1$

and $\sin4\theta=\sqrt{1-\left(2\left(T_2^*(x)\right)^2-1\right)^2}$. We find:

$$\Delta(x)=-\frac{8t^{4n+2}}{a^2c^2(t^4-1)}\left[\frac{\cos4n\theta-t^4\sin4n\theta\sin4\theta-t^4\cos4n\theta\cos4\theta}{4t^4\left[\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2\right]}+\frac{1}{t^8-1}\cos4n\theta\right]=$$

$$-\frac{8t^{4n+2}}{a^2c^2(t^4-1)(t^8-1)}\left[\frac{(t^8-1)\cos4n\theta-t^4\,(t^8-1)\sin4n\theta\sin4\theta-t^4\,(t^8-1)\cos4n\theta\cos4\theta}{4t^4\left[\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2\right]}+\cos4n\theta\right]=$$

$$-\frac{8t^{4n+2}}{a^2c^2(t^4-1)(t^8-1)}\left[\frac{(t^8-1)-t^4(t^8-1)\cos4\theta+4t^4\left[\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2\right]}{4t^4\left[\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2\right]}\cos4n\theta-\frac{t^4(t^8-1)\sin4\theta}{4t^4\left[\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2\right]}\sin4n\theta\right]=$$

$$-\frac{8t^{4n+2}}{a^2c^2(t^4-1)(t^8-1)}\left[\frac{(t^8-1)\left[1-t^4\left(2\left(T_2^*(x)\right)^2-1\right)+4t^4\left[\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2\right]\right]}{4t^4\left[\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2\right]}\cos 4n\theta -\right.$$

$$\left.\frac{t^4(t^8-1)\sqrt{1-\left(2\left(T_2^*(x)\right)^2-1\right)^2}}{4t^4\left[\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2\right]}\sin 4n\theta\right] =$$

$$= -\frac{8t^{4n+2}}{a^2c^2(t^4-1)(t^8-1)}\left[t^4\left[\frac{\left(2\frac{a^2}{c^2}-1\right)^2-(T_2^*(x))^2\left(2\left(2\frac{a^2}{c^2}-1\right)^2-1\right)}{\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2}\right]\cos 4n\theta -\right.$$

$$\left.\frac{2t^4\sqrt{\left(2\frac{a^2}{c^2}-1\right)^4-\left(2\frac{a^2}{c^2}-1\right)^2}\sqrt{\left(T_2^*(x)\right)^2-\left(T_2^*(x)\right)^4}}{\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2}\sin 4n\theta\right] \quad (6)$$

Now if we define:

$$h_1(x) := \frac{\left(2\frac{a^2}{c^2}-1\right)^2-(T_2^*(x))^2\left(2\left(2\frac{a^2}{c^2}-1\right)^2-1\right)}{\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2}$$

$$h_2(x) := \frac{2\sqrt{\left(2\frac{a^2}{c^2}-1\right)^4-\left(2\frac{a^2}{c^2}-1\right)^2}\sqrt{\left(T_2^*(x)\right)^2-\left(T_2^*(x)\right)^4}}{\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2}$$

noticing that:

$$h_1(-1) \le -1 \le h_1(x) \le 1 = h_1(0) \;\; ; x \in [-c, 0]$$
$$h_1(1) \le -1 \le h_1(x) \le 1 = h_1(0) \;\; ; x \in [0, c]$$

and

$$h_1^2(x) + h_2^2(x) = 1$$

we can easily say that for every $x \in [-c, c]$ there exists a $\lambda_x \in (0, \pi)$,so that:

$$\cos\lambda_x = h_1(x) = \frac{\left(2\frac{a^2}{c^2}-1\right)^2-(T_2^*(x))^2\left(2\left(2\frac{a^2}{c^2}-1\right)^2-1\right)}{\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2} \qquad (7)$$

$$\sin\lambda_x = h_2(x) = \frac{2\sqrt{\left(2\frac{a^2}{c^2}-1\right)^4-\left(2\frac{a^2}{c^2}-1\right)^2}\sqrt{\left(T_2^*(x)\right)^2-\left(T_2^*(x)\right)^4}}{\left(2\frac{a^2}{c^2}-1\right)^2-\left(T_2^*(x)\right)^2} \qquad (8)$$

Replacing (7) and(8) in (6) we obtain:

$$\Delta(x) = -\frac{8t^{4n+6}}{a^2c^2(t^4-1)(t^8-1)}[\cos\lambda_x\cos 4n\theta - \sin\lambda_x\sin 4n\theta]$$

$$= -\frac{8t^{4n+6}}{a^2c^2(t^4-1)(t^8-1)}\cos[4n\theta+\lambda_x]$$

Now if x varies from -c to 0, then θ varies from π to 0, x varies from 0 to $\frac{\pi}{2}$, λ_x varies from 0 to π and $\cos[4n\theta+\lambda_x]$ varies from $\cos n\pi$ to $\cos -\pi$. Hence when x varies from-c to c, $\cos[4n\theta+\lambda_x]$varies from $\cos 4n\pi$ to $\cos -\pi$ and consequently,

$$\cos[4n\theta+\lambda_x]$$

possesses at least $4n+2$ extremal points, where it assumes alternately the values $\pm\frac{8t^{4n+6}}{a^2c^2(t^4-1)(t^8-1)}$.

Therefore $p^*(x)$is the best approximation out of P_{4n} and

$\Delta(x) = -\frac{8t^{4n+6}}{a^2c^2(t^4-1)(t^8-1)}(-1)^k$; k=0,1,….,4n+1.

$$E_{4n}\left[\frac{1}{a^4-x^4},[-c,c]\right] = \|f-p_{4n}\|_\infty = \frac{8t^{4n+6}}{a^2c^2(t^4-1)(t^8-1)}$$

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