Weak Base and independent weak base of vector space

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Abstract

The problem of generation and oneness considered for expressing about an element is very important and has a big effect in mathematics in general and in algebra in special for example in vector spaces which has a finite dimension, where every element from this space is written in a lonely form in terms of elements of subset in this space and in this case we called this subset from the space (generated and linearly independent) base of space.

In second section of this paper we study the weak base for vector space, we obtained a full description weak dimension vector space and it is proved that finite weak bases for vector space have the same cardinality. In addition to that, it is proved that a finite subset X of vector space is weak base if and only if X is minimal weak generated set.

Also, we proved that every generated set of vector space contains weak base of this space. It is proved that for weak dimension spaces every weak independent subset can be extended to weak base for this space.

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In addition to that, we obtain many of important and interesting properties for weak base. In third section we study the independent weak bases for vector space and we proved that every subspace of some space is weak generated by subset not contained in it, contains weak independent base.
Finally, we proved the sufficient and necessary condition to be subspace U maximal in vector space V, that every weak independent base of subspace U is base of V. Also, we proved that many important and interesting properties for weak independent bases of vector space.

Key Words: Vector space, Generating and weakly generating, Linearly indep-endence and weakly independence, Base and Weak Base, independent weak base.

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القاعدة الضعيفة والقاعدة الضعيفة المستقلة لفضاء متجهي

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الملخص

تعد مسألتا التوليد ووحدانية كتابة عنصر من بنية جبرية كتركيب وحيد بدلالة عناصر من هذه البنية مهة جداً في البنى الجبرية. فعلى سبيل المثال، في الفضاءات المتجهية (الشعاعية) ذات البعد المنتهي، كل عنصر من الفضاء يكتب بشكل وحيد بدلالة عناصر مجموعة جزئية من هذا الفضاء، وفي هذه الحالة نسمي تلك المجموعة الجزئية من الفضاء (المولدة والمستقلة خطياً) بقاعدة الفضاء.

في الفقرة الثانية من هذه الورقة العلمية درسنا القواعد الضعيفة لفضاء متجهي منتهي البعد وقد حصلنا على التوصيف الكامل للفضاءات المتجهية ذات البعد الضعيف، وقد تم إثبات أن القواعد المنتهية الضعيفة لفضاء متجهي لها القدرة ذاتها. فضلاً عن ذلك، تم إثبات أن القاعدة المنتهية الضعيفة لفضاء متجهي هي وفقط هي المجموعة المولدة الضعيفة الأصغرية، كما أن أي مجموعة مولدة ضعيفة لفضاء متجهي تحوي قاعدة ضعيفة لهذا الفضاء. أيضاً تم إثبات أنه في الفضاءات ذات البعد المنتهي

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الضعيف كل مجموعة جزئية مستقلة بضعف يمكن تمديدها إلى قاعدة ضعيفة لهذا الفضاء. إضافة لذلك حصلنا على عدد من الخصائص المهمة للقواعد الضعيفة.
وفي الفقرة الثالثة درسنا القواعد الضعيفة المستقلة لفضاء متجهي جزئي وقد أثبتنا أن كل فضاء جزئي من فضاء ما مولدة بضعف بمجموعة جزئية غير محتواة فيه، يحوي قاعدة ضعيفة مستقلة. فضلاً عن ذلك، تم إثبات أن الشرط اللازم والكافي كي يكون U فضاء جزئياً أعظمياً من فضاء متجهي V هو أن تكون كل قاعدة ضعيفة مستقلة للفضاء الجزئي U وغير محتواة في V هي قاعدة للفضاء V. كما تم في هذه الفقرة إثبات العديد من الخصائص المهمة للقواعد الضعيفة المستقلة لفضاء متجهي جزئي.

الكلمات المفتاحية: فضاء متجهي، التوليد والتوليد الضعيف، الاستقلال الخطي والاستقلال الخطي الضعيف، القاعدة والقاعدة الضعيفة، القاعدة الضعيفة المستقلة.

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1. Introduction:

The problem of generation and oneness considered for expressing about an element is very important and has a big effect in mathematics in general and in algebra in special, for example in vector spaces which has a finite dimension, every element from this space is written in one and only one way as linear combination of elements of subset in this space and in this case we called this subset from the space (generated and linearly independent) base of space. The role of base of vector space effects the minds of many algebraist internationality and they try to generalize the base concept to other algebraic structures entitled the weak base and the first attempt is the looking for the smallest generated sets for many algebraic structures as group, ring and others which is studied by L. Halbeisen in [6], [7] and [10].

In 2014 Michal Hrbek in [4], generalize the concept of bases of modules, where he define a weak base (generated and weakly independent) of module M over an associative and unitary ring R, where the subset X of M is weak base of M, if X generated this module and satisfy the following condition, for any finite set of elements $x_1, x_2, \cdots, x_n \in X$ such that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$ where $\alpha_1, \alpha_2, \cdots, \alpha_n \in R$ follows that all elements α_i $(1 \le i \le n)$ not invertible in R. In his work he fined the sufficient and necessary condition for some module over Dedekind to have weak base.

In 2015, R. Pavel continues the work of H. Michal in [5], where he describe Abelian group which has a minimal generated set and he fined the sufficient and necessary condition for Abelian group to have a minimal generated set.

In 2016, R. Pavel and H. Daniel in [3] answered positively the following question is the module over finite product of divisible rings having weak base.?

In [1] we study the concept of weak generation of vector spaces and study weak independence. In this work we continue our study of this concepts throw study weak independent and weak bases of vector spaces.

In second section of this paper we study the weak base for vector space, we obtained a full description weak dimension vector space and it is proved that finite weak bases for vector space have the same cardinality. In addition to that, it is proved that a finite subset X of vector space is weak base if and only if X is minimal weak generated set.

Also, we proved that every generated set of vector space contains weak base of this space. It is proved that for weak dimension spaces every weak independent subset can be extended to weak base for this space.

In addition to that, we obtain many of important and interesting properties for weak base. In third section we study the independent weak bases for vector space and we proved that every subspace of some space is weak generated by subset not contained in it, contains weak independent base.

Finally, we proved the sufficient and necessary condition to be subspace U maximal in vector space V, that every weak independent base of subspace U is base of V. Also, we proved that many important and interesting properties for weak independent bases of vector space.

2. Weak Bases and Weak Dimension.

In this section we study the notion of a weak base of a finite dimensional vector space over a field and its basic properties. We start with the following definition:

Definition [1]. A finite subset $X=\{v_1,v_2,\cdots,v_n\}$ of a vector space V over a filed F is *weakly generated* of V over F, if for every element $x\in V$ there exists $\{\alpha_i\}_{i=1}^{n}\subset F$ such that $x=\sum_{i=1}^{n}\alpha_i v_i$ and $\sum_{i=1}^{n}\alpha_i=0$, in this case we write $V=\langle X\rangle_W$.

Lemma 2.1 [1]. Let V be a vector space over a field F and X,Y are finite subsets of V. Then the following are hold:

1 – $\langle v\rangle_W=\{0\}$ for every element $v\in V$.

2 – If $\langle X\rangle_W\neq\{0\}$, then $Card\ X\geq 2$.

3 – $\langle X\rangle_W\subseteq\langle X\rangle$.

4 – $\langle X\rangle_W=\langle X\rangle$ if and only if $X\subseteq\langle X\rangle_W$.

5 – If X is linearly independent, then $X\not\subset\langle X\rangle_W$ and $\langle X\rangle\neq\langle X\rangle_W$.

6 – If $X \subseteq Y$, then $\langle X\rangle_W \subseteq \langle Y\rangle_W$.

7 – If $X \subseteq \langle Y\rangle_W$, then $\langle X\rangle_W \subseteq \langle Y\rangle_W$ and $\langle X\rangle \subseteq \langle Y\rangle_W$.

8 – If $0 \notin X$, then $V = \langle X\rangle$ if and only if $V = \langle X \cup \{0\}\rangle_W$.

Definition [1]. A finite subset $X = \{v_1, v_2, \cdots, v_n\}$ of a vector space V over a filed F is *weakly independent* in V if, for any $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ such that $\sum_{i=1}^{n} \alpha_i v_i = 0$ and $\sum_{i=1}^{n} \alpha_i = 0$, implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

If X is not weakly independent in V, then we say that X is *fully dependent*.

Lemma 2.2 [1]. Let V be a vector space over a field F. The following hold:

1 – The set $\{0\}$ is weakly independent in V.

2– Each subset of V consisting of two different elements is weakly independent. In particular, for each non-zero element $v \in V$, $\{0, v\}$ is weakly independent.

3 – Each independent finite subset of V is weakly independent.

4 – Let $X = \{v_1, v_2, \cdots, v_n\}$ be an independent subset of V, then for any $v \in V$, the set $Y = \{v_1 - v, v_2 - v, \cdots, v_n - v\}$ is weakly independent.

5 – If $0 \notin X$, then X is independent if and only if $X \cup \{0\}$ is weakly independent.

6 – Let $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V, then X is fully dependent if and only if there exists an element $v_j \in X$, $1 \le j \le n$ for which there are

$\alpha_1, \alpha_2, \cdots, \alpha_{j-1}, \alpha_{j+1}, \cdots, \alpha_n \in F$

Such that $v_j = \sum_{i=1,\, i \ne j}^{n} \alpha_i v_i$ and $\sum_{i=1,\, i \ne j}^{n} \alpha_i = 1$.

Definition [8]. A finite subset $X=\{v_1,v_2,\cdots,v_n\}$ of a vector space V is called linearly independent if for every $\alpha_1,\alpha_2,\cdots,\alpha_n\in F$ such that $\sum_{i=1}^{n}\alpha_i v_i=0$, then $\alpha_i=0$ for all $1\le i\le n$.

Definition [2]. A finite subset X of a vector space V is called a *base* of V, if it is generates V, i.e., $V=\langle X\rangle$ and is linearly independent.

Definition [9]. If a finite subset X of a vector space V is base of V, then $Card\ X$ is called dimension of V.

Definition. Let V be a vector space over a field F and X be a finite subset of V. We say that X is a *weak base* of V, if it satisfies the following:

1 – X is weakly generated of V, i.e., $V=\langle X\rangle_w$.

2 – X is weakly independent.

Now, we provide the following Lemma which is useful in the proof of further Theorem;

Lemma 2.3 [1]. Let V be a vector space over a field F and X be a finite subset of V. The following are equivalent:

1 – X is weakly independent.

2 – The zero element of V is written as a weak linear combination of X as the only form.

3 –Each $v\in\langle X\rangle_w$ is written as a weak linear combination of X as the only form.

Theorem 2.4. Let V be a vector space over a field F, and X be a finite subset of V. The following are equivalent:

1 – X is a weak base of V.

2 – Each $v\in V$ is written as a weak linear combination of X as the only form.

Proof. $(1)\Rightarrow(2)$. Suppose that X is a weak base of V. Then X is weakly independent and each element $v\in V$ is written as a weak linear combination of X as the only form by Lemma 2.3.

$(2)\Rightarrow(1)$. Obviously $V=\langle X\rangle_w$. Also, X is weakly independent over F by Lemma 2.3. Therefore, X is a weak base of V.

Theorem 2.5. Let V be a vector space over a field F, and $X=\{v_1,v_2,\cdots,v_n\}$ be a subset of V. If X is a weak base of V, then any subset of V which consists of m elements where $m>n$, is fully dependent.

Proof. Suppose that $X=\{v_1,v_2,\cdots,v_n\}$ is a weak base of V, and let $Y=\{u_1,u_2,\cdots,u_m\}$ be a finite subset of V, where $m>n$. Since X is a weak base of V, then for every $u_j\in Y$ $(1\le j\le m)$ there exist elements $a_{1j},a_{2j},\cdots,a_{nj}\in F$ such that:

$u_j=\sum_{i=1}^{n}a_{ij}v_i$, and $\sum_{i=1}^{n}a_{ij}=0$.

Let $b_1,b_2,\cdots,b_m\in F$ such that:

$\sum_{i=1}^{m}b_iu_i=0$, and $\sum_{i=1}^{m}b_i=0$.

Then:

$$(b_1a_{11})v_1+(b_1a_{21})v_2+(b_1a_{31})v_3+\cdots+(b_1a_{n1})v_n+$$
$$+(b_2a_{12})v_1+(b_2a_{22})v_2+(b_2a_{32})v_3+\cdots+(b_2a_{n2})v_n+$$
$$\vdots$$
$$+(b_ma_{1m})v_1+(b_ma_{2m})v_2+(b_ma_{3m})v_3+\cdots+(b_ma_{nm})v_n=0.$$

Consequently, we find that:

$$(b_1a_{11}+b_2a_{12}+b_3a_{13}+\cdots+b_ma_{1m})v_1+$$
$$+(b_1a_{21}+b_2a_{22}+b_3a_{23}+\cdots+b_ma_{2m})v_2+$$
$$\vdots$$
$$+(b_1a_{n1}+b_2a_{n2}+b_3a_{n3}+\cdots+b_ma_{nm})v_n=0$$

and that:

$$(b_1a_{11}+b_2a_{12}+b_3a_{13}+\cdots+b_ma_{1m})+(b_1a_{21}+b_2a_{22}+b_3a_{23}+\cdots+b_ma_{2m})+$$
$$+\cdots+(b_1a_{n1}+b_2a_{n2}+b_3a_{n3}+\cdots+b_ma_{nm})=$$
$$b_1(a_{11}+a_{21}+a_{31}+\cdots+a_{n1})+b_2(a_{12}+a_{22}+a_{32}+\cdots+a_{n2})+$$
$$+\cdots+b_m(a_{1m}+a_{2m}+a_{3m}+\cdots+a_{nm})=\sum_{j=1}^{m}b_j(\sum_{i=1}^{n}a_{ij})=\sum_{j=1}^{m}b_j0=0$$

Since X is weakly independent, we find that:

$$b_1a_{11}+b_2a_{12}+b_3a_{13}+\cdots+b_ma_{1m}=0$$
$$b_1a_{21}+b_2a_{22}+b_3a_{23}+\cdots+b_ma_{2m}=0 \vdots$$
$$b_1a_{n1}+b_2a_{n2}+b_3a_{n3}+\cdots+b_ma_{nm}=0$$
$$b_1+b_2+b_3+\cdots+b_m=0$$

The last system of $n+1$ homogeneous linear equations in m unknowns, is equivalent to:

$$b_1a_{21}+b_2a_{22}+b_3a_{23}+\cdots+b_ma_{2m}=0 \vdots$$
$$b_1a_{n1}+b_2a_{n2}+b_3a_{n3}+\cdots+b_ma_{nm}=0$$
$$b_1+b_2+b_3+\cdots+b_m=0$$

Since $m>n$, then the number of equations is less than the number of unknowns. Therefore, the above homogeneous linear equation system has an infinite number of solutions. This shows that $Y=\{u_1,u_2,\cdots,u_m\}$ is fully dependent.

Theorem 2.6. Let V be a vector space over a field F, and $X=\{v_1,v_2,\cdots,v_n\}$ be a subset of V. If X is a weak base of V, then any other weak base of V also consists of n elements.

Proof. Suppose that $Y=\{u_1,u_2,\cdots,u_m\}$ is another weak base V. We suppose that $m\neq n$, then we can recognize two cases:

1 – If $m>n$, then X is fully dependent by Theorem 2.5, a contradiction.

2 – If $m<n$, then Y is fully dependent by Theorem 2.5, a contradiction.

Therefore, $m=n$.

If a finite subset X of V is a weak base of V, then we call $Card\ X$ a *weak dimension* of V over F, and we denote it as $w\cdot\dim_F V=Card\ X$. In this case, we say that V is a *finite weak dimensional vector space*.

Example. With R as the field of real numbers, let $X = \{(1,0),(0,1),(2,3)\}$ be a subset of the vector space R^2 over R. It's easy to show that X is weakly independent, and $R^2 = \langle X \rangle_W$, where:

$$(x,y) = \frac{x-y}{2}(1,0) + \frac{y-3x}{4}(0,1) + \frac{x+y}{4}(2,3)$$

for any $(x,y) \in R^2$. Thus, X is a weak base of R^2, and $w\cdot\dim_R R^2 = 3$.

Lemma 2.7. Let V be a vector space over a field F. The following hold:

1 – $w\cdot\dim_F \langle\{0\}\rangle = 1$, i.e., $\{0\}$ is a weak base of the zero vector space.

2 – If $V \neq \{0\}$, and X is a weak base of F, then:

$Card\ X = w\cdot\dim_F V \geq 2$.

3 – If X is a weak base of V, then $V = \langle X \rangle$ and $w\cdot\dim_F V \geq 1$.

Proof. 1 – We have $\langle\{0\}\rangle_W = \{0\}$ by Lemma 2.1(1), and the set $\{0\}$ is weakly independent by Lemma 2.2(1). Therefore $\{0\}$ is a weak base of the zero vector space, and $w\cdot\dim_F \langle\{0\}\rangle = 1$.

2 – Direct by Lemma 2.1(4), and (1).

3 – Suppose that X is a weak base of V, then $V = \langle X \rangle_W$. Since $X \subseteq V$, then $V = \langle X \rangle$ by Lemma 2.1(4). Also $w\cdot\dim_F V \geq 1$ by (1) and (2).

Now, we provide the following Lemma which is useful in the proof of following important Theorem:

Lemma 2.8 [1]. Let V be a vector space over a field F and X be a finite subset of V. Then the following hold:

1 – If X is a minimal weakly generated set of V ,then X is weakly independent.

2 – If X is maximal weakly independent, then X is a minimal weakly generated set of V .

Theorem 2.9. Let V be a vector space over a field F, and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V . The following are equivalent:

1 – X is a weak base of V .

2 – X is maximal weakly independent.

3 – X is minimal weakly generated set of V .

Proof. $(1) \Rightarrow (2)$. Suppose that X is a weak base of V . Let $v \in V$ such that $v \notin X$, since $Card\ (X \cup \{v\}) = n+1 > n$, then $X \cup \{v\}$ is fully dependent by Theorem 2.5. Therefore X is maximal weakly independent.

$(2) \Rightarrow (3)$. Direct by Lemma 2.8.

$(3) \Rightarrow (1)$. Suppose that X is minimal weakly generated set of V , then X is weakly independent by Lemma 2.8. Therefore X is a weak base of V .

Theorem 2.10. Let V be a vector space over a field F, and X be a finite subset of V such that $0 \notin X$. The following are equivalent:

1 – X is a base of V .

2 – $X \cup \{0\}$ is a weak base of V .

Proof. $(1) \Rightarrow (2)$. Suppose that X is a base of V, since X is independent, then $X \cup \{0\}$ is weakly independent by Lemma 2.2(5). On other hand, X generates V , then $X \cup \{0\}$ is weakly generated of V by Lemma 2.1(8). Therefore $X \cup \{0\}$ is a weak base of V .

$(2) \Rightarrow (1)$. Suppose that $X \cup \{0\}$ is a weak base of V, since $X \cup \{0\}$ is weakly independent, then X is independent by Lemma 2.2(5). On other hand, $X \cup \{0\}$ is weakly generated V, then X generates V by Lemma 2.1(8). Therefore X is a base of V .

According to the last Theorem, we can form the following Lemma:

Lemma 2.11. Let V be a vector space over a field F and $dim_F V = n$. The following hold:

1 – $w \cdot dim_F V = n+1$.

2 – Let V_1 and V_2 are subspaces of V such that $V = V_1 + V_2$, then:
$w \cdot dim_F V = w \cdot dim_F V_1 + w \cdot dim_F V_2 - w \cdot dim_F (V_1 \cap V_2)$
3 – Let V_1 and V_2 are subspaces of V such that $V = V_1 \oplus V_2$, then:
$w \cdot dim_F V = w \cdot dim_F V_1 + w \cdot dim_F V_2 - 1$
4 – Let $\{V_i\}_{i=1}^{k}$ be a family of subspaces of V such that $V = \sum_{i=1}^{k} \oplus V_i$, then:
$w \cdot dim_F V = \sum_{i=1}^{k} w \cdot dim_F V_i - (k-1)$

Proof. 1 – Since $dim_F V = n$, then any base of V consists of n elements. Let X be a base of V, then $X \cup \{0\}$ is a weak base of V by Theorem 2.10. On the other hand, any other weak base of V also consists of $n+1$ elements by Theorem 2.6. Therefore, $w \cdot dim_F V = n+1$.
2 – Since $V = V_1 + V_2$, then:
$dim_F V = dim_F V_1 + dim_F V_2 - dim_F (V_1 \cap V_2)$.
Then:
$w \cdot dim_F V = w \cdot dim_F V_1 + w \cdot dim_F V_2 - w \cdot dim_F (V_1 \cap V_2)$
by (1).
3 – Since $V = V_1 \oplus V_2$, then $V = V_1 + V_2$ and $V_1 \cap V_2 = \{0\}$. Then by (2) implies:
$w \cdot dim_F V = w \cdot dim_F V_1 + w \cdot dim_F V_2 - w \cdot dim_F (\{0\})$
On the other hand, $w \cdot dim_F \langle \{0\} \rangle = 1$ by Lemma 2.7(1). Therefore:
$w \cdot dim_F V = w \cdot dim_F V_1 + w \cdot dim_F V_2 - 1$
4 – Direct by (3) and induction on n.
Now, we provide the following Lemma which is useful in the proof of following important Theorem:
Lemma 2.12 [1]. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. Then for any element $v_0 \in \langle X \rangle$ such that $v_0 = \sum_{i=1}^{n} a_i v_i$ and

$\sum_{i=1}^{n} a_i \neq 1$ where $a_1, a_2, \cdots, a_n \in F$, the following hold:

1 – If X is independent, then $X \cup \{v_0\}$ is weakly independent.

2 – If X is maximal independent, then $X \cup \{v_0\}$ is maximal weakly independent.

3 – If X is maximal weakly independent, then X is dependent.

4 – If X is maximal weakly independent, then there exists an element $v_j \in X$ $(1 \leq j \leq n)$ can be written as a linear combination of $X \setminus \{v_j\}$, such that $X \setminus \{v_j\}$ is maximal independent.

Moreover, $v_j = \sum_{i=1,\, i \neq j}^{n} \beta_i v_i$ and $\sum_{i=1,\, i \neq j}^{n} \beta_i \neq 1$ where $\{\beta_i\}_{i=1, i \neq j}^{n} \subset F$.

Theorem 2.13. Let V be a vector space over a field F, and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V**.** The following hold:

1 – If X is a base of V, then X can be expanded to a weak base of V.

2 – If X is a weak base of V, then there exist an element $v_j \in X$, $(1 \leq j \leq n)$ such that $X \setminus \{v_j\}$ is a base of V, also:

$v_j = \sum_{i=1,\, i \neq j}^{n} \alpha_i v_i$, and $\sum_{i=1,\, i \neq j}^{n} \alpha_i \neq 1$

where $\{\alpha_i\}_{i=1, i \neq j}^{n} \subset F$.

3 – There exist a finite subset Y of V, such that Y is a weak base of V.

Proof. 1 – Suppose that X is a base of V, then the set X is maximal independent by Theorem 2.9. Let $v \in V$ such that $v = \sum_{i=1}^{n} \alpha_i v_i$ where $\alpha_i \in F$ for $1 \leq i \leq n$, and $\sum_{i=1}^{n} \alpha_i \neq 1$. Then the set $X \cup \{v\}$ is maximal weakly independent by Lemma 2.12(2). Therefore, $X \cup \{v\}$ is a weak base of V by Theorem 2.9.

2 – Suppose that X is a weak base of V, then X is maximal weakly independent by Theorem 2.9. So, X is dependent by Lemma

2.12(3). Hence, there exist an element $v_j \in X$ ($1 \le j \le n$) can be written as a linear combination of $X \setminus \{v_j\}$, then $X \setminus \{v_j\}$ is maximal independent by Lemma 2.12 (4). Therefore, $X \setminus \{v_j\}$ is a base of V. On other hand:

$v_j = \sum_{i=1,\, i\ne j}^{n} \alpha_i v_i$, and $\sum_{i=1,\, i\ne j}^{n} \alpha_i \ne 1$

where $\{\alpha_i\}_{i=1, i\ne j}^{n} \subset F$ by Lemma 2.12(4).

3 – Direct by (1), since every vector space has a base.

Theorem 2.14. Let V be a vector space over a field F, and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. The following are equivalent:

1 – X is weakly generated of V, i.e., $V = \langle X \rangle_W$.

2 – X contains a minimal weakly generated set of V.

3 – X contains a weak base of V.

Proof. $(1) \Rightarrow (2)$. Suppose that $V = \langle X \rangle_W$. If X is weakly independent, then X is a weak base of V, then X is a minimal weakly generated set of V by Theorem 2.9. Suppose that X is not weakly independent, then X is fully dependent, then by Lemma 2.2(6), there exists an element let it be $v_1 \in X$ such that $v_1 = \sum_{i=2}^{n} \alpha_i v_i$ and $\sum_{i=2}^{n} \alpha_i = 1$ where $\alpha_i \in F$ $(2 \le i \le n)$. It is obvious that $Y_1 = X \setminus \{v_1\}$ is weakly generated of V. If Y_1 is weakly independent, then Y_1 is a weak base of V, then Y_1 is a minimal weakly generated set of V by Theorem 2.9, and $Y_1 \subset X$. Suppose that Y_1 is not weakly independent, then Y_1 is fully dependent, then by Lemma 2.2(6), there exists an element let it be $v_2 \in Y_1$ such that $v_2 = \sum_{i=3}^{n} \beta_i v_i$ and $\sum_{i=3}^{n} \beta_i = 1$ where $\beta_i \in F$ $(3 \le i \le n)$. It is obvious that $Y_2 = Y_1 \setminus \{v_2\}$ is weakly generated of V. If Y_2 is weakly independent, then Y_2 is a weak base of V, then Y_2 is a minimal weakly generated set of V by Theorem 2.9, and $Y_2 \subset X$. By proceeding this way we get a set:

$Y = X \setminus \{v_1, v_2, \cdots, v_i\}; (1 \le i < n)$

where Y is weakly independent, and is weakly generated of V, then Y is a weak base of V, then Y is a minimal weakly generated set of V by Theorem 2.9, and $Y \subset X$. Therefore X contains a minimal weakly generated set of V.

$(2) \Rightarrow (3)$. Suppose that $Y \subseteq X$ is a minimal weakly generated set of V, then Y is a weak base of V by Theorem 2.9.

$(3) \Rightarrow (1)$. Without loss of generality, suppose that $Y = \{v_1, v_2, \cdots, v_m\} \subseteq X$ where $m \le n$ is a weak base of V, then for any $v \in V$ then there exist elements $a_1, a_2, \cdots, a_m \in F$ such $v = \sum_{i=1}^{m} \alpha_i v_i$, and $\sum_{i=1}^{m} \alpha_i = 0$. Then, for:

$\alpha_{m+1} = \alpha_{m+2} = \cdots = \alpha_n = 0$

we find that $v = \sum_{i=1}^{n} \alpha_i v_i$, and $\sum_{i=1}^{n} \alpha_i = 0$, then X is weakly generated of V, i.e., $V = \langle X \rangle_W$.

Theorem 2.15. Let V be a vector space over a field F, and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. If $w.dim_F V = n$, then the following hold:

1 – If X is weakly independent, then X is a weak base of V.

2 – If X is weakly generated of V, then X is a weak base of V.

Proof. 1 – Suppose that X is weakly independent. Since $w.dim_F V = n$, then any subset of V which consists of $n+1$ elements is fully dependent by Theorem 2.5. Thus, X is maximal weakly independent. Therefore, X is a weak base of V by Theorem 2.9.

2 – Suppose that X is weakly generated of V. Then X contains a weak base of V by Theorem 2.14, and let it be S .Since $S \subseteq X$, then $Card\, S \le Card\, X$, where both X and S are finite sets. We

suppose that $Card\ S \neq Card\ X$, then $Card\ S < n$, i.e., $w.dim_F\ V < n$, a contradiction. Thus:

$Card\ S = Card\ X = n$

Since $S \subseteq X$, then $S = X$. Therefore, X is a weak base of V.

Theorem 2.16. Let V be a vector space over a field F, and $w.dim_F\ V = n$, then any weakly independent finite subset of V can be expanded to a weak base of V.

Proof. Let $X = \{v_1, v_2, \cdots, v_m\}$ be a weakly independent subset of V. Since $w.dim_F\ V = n$, then $m \leq n$ by Theorem 2.5.

– If $m = n$, then X is a weak base of V by Theorem 2.15.

– If $m < n$, then we recognize two cases:

I – If X independent , then X can be expanded to a base of V, let it be Y. On the other hand since Y is a base of V, then Y can be expanded to a weak base of V by Theorem 2.13.

II – If X is weakly independent and non-independent, then there exists an element and let it be $v_1 \in X$ can be written as a linear combination of $X \setminus \{v_1\}$. Thus, $v_1 = \sum_{i=2}^{m} \alpha_i v_i$ and $\sum_{i=2}^{m} \alpha_i \neq 1$ where $\alpha_i \in F$ for all $2 \leq i \leq m$ by [1 , Theorem 3.11]. Also $X \setminus \{v_1\}$ is independent by [1 , Theorem 3.9]. Therefore, $X \setminus \{v_1\}$ can be expanded to a base of V, let it be Y. Since Y is a base of V, and $X \setminus \{v_1\} \subseteq Y$, then Y is maximal independent, and $v_1 \notin Y$. Thus, $Y \cup \{v_1\}$ is maximal weakly independent by Lemma 2.12(2). Therefore, $Y \cup \{v_1\}$ is a weak base of V by Theorem 2.9.

Theorem 2.17. Let V be a vector space over a field F, and $w.dim_F\ V = n$, then for any subspace K of V the following are hold:

1 – $w \cdot dim_F K \leq w \cdot dim_F V$.

2 – Any weak base of K can be expanded to a weak base of V.

3 – $K = V$ if and only if $w \cdot dim_F K = w \cdot dim_F V$.

Proof. 1 – Since K subspace of V, then $dim_F K \leq dim_F V$. Hence:

$1+dim_F K \leq 1+dim_F V$

Thus $w\cdot dim_F K \leq w\cdot dim_F V = n$ by Lemma 2.11.

2 – Suppose that Y is a weak base of K, then Y is weakly independent. Therefore, Y can be expanded to a weak base of V by Theorem 2.16.

3 – ($\Rightarrow$) If $K = V$, then $w\cdot dim_F K = w\cdot dim_F V$.

($\Leftarrow$) Suppose that $w\cdot dim_F K = w\cdot dim_F V$, then then there exists a weak base of K consists of n elements, let it be X, then $K = \langle X \rangle_W$ and X is weakly independent. On other hand, since $X \subseteq K \subseteq V$, and $w\cdot dim_F V = n$, then X is a weak base of V by Theorem 2.15. Therefore $V = \langle X \rangle_W = K$.

3. Independent weak base of a subspace.

Let V be a vector space over a field F and X be an independent finite subset of V, then X is weakly independent by Lemma 2.2(3). Moreover, $X \not\subset \langle X \rangle_W$ by Lemma 2.1(5). Thus, X is not a weak base of $\langle X \rangle_W$. In this section, we study a weakly generated subspace U by an independent finite subset of V. We start with the following lemma:

Lemma 3.1. Let V be a vector space over a field F, and U be a subspace of V.

Suppose that X, Y are finite subsets of V such that $U = \langle X \rangle_W = \langle Y \rangle_W$. If $X \subseteq Y$ and $Y \not\subset U$, then $X \not\subset U$.

Proof. Suppose that the set $Y = \{v_1, v_2, \cdots, v_n\}$ is weakly generated of U, i.e., $U = \langle Y \rangle_W$, and $Y \not\subset U$. Let $X = \{v_{i1}, v_{i2}, \cdots, v_{im}\}$ be a subset of Y such that $U = \langle X \rangle_W$ where $n \geq m$. We suppose that $X \subseteq U$, then $U = \langle X \rangle$ by Lemma 2.1(4). On the other hand, since

$X \subseteq Y$ then, $U \subseteq \langle Y \rangle$. Let $w \in \langle Y \rangle$, then there exist elements $\beta_1, \beta_2, \cdots, \beta_n \in F$ such that $w = \sum_{i=1}^{n} \beta_i v_i$. Also, $v_{ij} \in Y \subseteq \langle Y \rangle$ for each $v_{ij} \in X$ where $1 \leq j \leq m$. Consequently:

$$w' = w - (\sum_{k=1}^{n} \beta_k) v_{ij} \in \langle Y \rangle$$

Since $(\sum_{k=1}^{n} \beta_k) - (\sum_{k=1}^{n} \beta_k) = 0$, then $w' \in \langle Y \rangle_W = U$. As long as $U = \langle X \rangle$ there exist elements $\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{im} \in F$ such that $w' = \sum_{j=1}^{m} \alpha_{ij} v_{ij}$. Hence:

$$w = \sum_{j=1}^{m} \alpha_{ij} v_{ij} + (\sum_{k=1}^{n} \beta_k) v_{ij}$$

Thus, $w \in U$, i.e., $\langle Y \rangle \subseteq U$. Thus, $U = \langle Y \rangle$, then $Y \subseteq U$ by Lemma 2.1(4), a contradiction. Therefore, $X \not\subset U$.

Theorem 3.2. Let V be a vector space over a field F, and $X = \{v_1, v_2, \cdots, v_n\}$ be an independent subset of V. The following hold:

1 – $w.dim_F \langle X \rangle_W = n$, and $dim_F \langle X \rangle_W = n-1$.

2 – The set:

$$X_i = \{v_1 - v_i, v_2 - v_i, \cdots, v_{i-1} - v_i, 0, v_{i+1} - v_i, \cdots, v_n - v_i\} \subseteq \langle X \rangle_W$$

is a weak base of $\langle X \rangle_W$ for all $1 \leq i \leq n$. Also $X_i \setminus \{0\}$ is a base of $\langle X \rangle_W$.

Proof. 1 – Suppose that X is independent, then $X \not\subset \langle X \rangle_W$ and $\langle X \rangle \neq \langle X \rangle_W$ by Lemma 2.1(5).

On the other hand, since $\langle X \rangle_W \subset \langle X \rangle$ by Lemma 2.1(3) and $dim_F \langle X \rangle = n$, then $dim_F \langle X \rangle_W < n$. Thus, $w.dim_F \langle X \rangle_W < n+1$ by Lemma 2.11(1), Hence:

$$w.dim_F \langle X \rangle_W \leq n$$

Then, the cardinality of any weak base of $\langle X \rangle_W$ is not greater than n by Theorem 2.6. On the other hand, the set:
$Y = \{0, v_2 - v_1, \cdots, v_n - v_1\} \subseteq \langle X \rangle_W$
is weakly independent by Lemma 2.2(4). Since $Card\, Y = n$, then Y is maximal weakly independent subset of $\langle X \rangle_W$. Hence, Y is a weak base of $\langle X \rangle_W$ by Theorem 2.9. Therefore, $w.dim_F \langle X \rangle_W = n$. Thus, $dim_F \langle X \rangle_W = n-1$ by Lemma 2.11.

2 – Let $v_i \in X$, where $1 \le i \le n$, then the set:
$X_i = \{v_1 - v_i, v_2 - v_i, \cdots, v_{i-1} - v_i, 0, v_{i+1} - v_i, \cdots, v_n - v_i\} \subseteq \langle X \rangle_W$
is weakly independent by Lemma 2.2(4). Since $w.dim_F \langle X \rangle_W = n$ by (1), and since $Card\ X_i = n$ then, X_i is a weak base of $\langle X \rangle_W$ by Theorem 2.15. Also, $X_i \setminus \{0\}$ is a base of $\langle X \rangle_W$ by Theorem 2.10.

Now, we state the concept of an independent weak base of a subspace and its properties, we start with the following definition:

Definition. Let V be a vector space over a field F, and U be a subspace of V. We say that the finite subset X of V is an independent weak base of U, if it satisfies the following:

1 – X is weakly generated of U, i.e., $U = \langle X \rangle_W$.

2 – X is independent.

Lemma 3.3. Let V be a vector space over a field F. The following hold:

1 – If the finite subset X of V is an independent weak base of a subspace U, then $w \cdot \dim_F U = Card\ X$ and $X \not\subset U$.

2 – For any non-zero element $v \in V$ the set $\{v\}$ is an independent weak base of the zero subspace.

3 – If X is a weakly independent finite subset of V, then:
$w \cdot dim_F \langle X \rangle_W = Card\, X$.

Proof. 1 – Direct by Theorem 3.2, and Lemma 2.1(5).
2 – Let $v \in V$ be a non-zero element, then $\langle \{v\} \rangle_W = 0$ by Lemma 2.1(1). Therefore, $\{v\}$ is an independent weak base of the zero subspace of V, where $\{v\}$ is an independent subset of V.
3 – Suppose that X is a weakly independent finite subset of V, then we recognize two cases:
– X is weakly independent and non-independent, then X is a weak base of $\langle X \rangle_W$. Thus, $w \cdot dim_F \langle X \rangle_W = Card\, X$.
– X is independent, then X is an independent weak base of $\langle X \rangle_W$. Thus:

$$w \cdot dim_F \langle X \rangle_W = Card\, X$$

by (1).

Example. With R as the field of real numbers, let $R_2[x]$ the vector space of polynomials over R of degrees at most 2 in indeterminate x and $R_1[x]$ the vector space of polynomials over R of degrees at most 1 in indeterminate x. $R_1[x]$ is a subspace of $R_2[x]$. Moreover, the set:

$$X = \{-1+x^2, 1+x^2, x+x^2\} \subseteq R_2[x]$$

is an independent weak base of $R_1[x]$, where:

$$ax+b = \frac{-a-b}{2}(-1+x^2) + \frac{-a+b}{2}(1+x^2) + a(x+x^2)$$

for every $ax+b \in R_1[x]$; $a, b \in R$.

Theorem 3.4. Let V be a vector space over a field F, and U be a subspace of V. Suppose that X is an independent finite subset of V. The following are equivalent:
1 – X is an independent weak base of U.
2 – Each $u \in U$ is written as a weak linear combination of X as the only form.

Proof. $(1) \Rightarrow (2)$. Suppose that X is an independent weak base of U, then $U = \langle X \rangle_W$. Since X is independent, then X is weakly independent by Lemma 2.2(3). Therefore, each $u \in U$ is written as a weak linear combination of X as the only form by Lemma 2.3.
$(2) \Rightarrow (1)$. It's obvious that X is weakly generated of U. Since X is independent, then X is an independent weak base of U.
Theorem 3.5. Let V be a vector space over a field F, and U be a subspace of V. Suppose that $X = \{v_1, v_2, \cdots, v_n\}$ is an independent weak base of U. The following hold:
1 – Any finite subset Y of V consisting of m where $m > n$ such that $Y \not\subset U$ and $U = \langle Y \rangle_W$, is fully dependent.
2 – Any independent weak base of U consists of n elements.
3 – X is the smallest finite subset of V such that $X \not\subset U$ and $U = \langle X \rangle_W$, i.e., if Y is a finite subset of V such that $Y \not\subset U$ and $U = \langle Y \rangle_W$, then $Card\, Y \geq n$.
4 – Any independent finite subset Z of V such that $U = \langle Z \rangle_W$, consists of n elements.
Proof. 1 – Suppose that X is an independent weak base of U, then $w \cdot \dim_F U = n$ by Lemma 3.3. Let $Y = \{u_1, u_2, \cdots, u_m\}$ where $m > n$ be a finite subset of V such that $Y \not\subset U$ and $U = \langle Y \rangle_W$, then $u_i - u_1 \in U$ for any $1 \leq i \leq m$, so that $Y_1 = \{0, u_2 - u_1, \cdots, u_m - u_1\} \subseteq U$. Since $m > n$, then Y_1 is fully dependent by Theorem 2.5. Thus, there exist elements $a_1, a_2, \cdots, a_m \in F$ not all zero such that:
$\sum_{i=1}^{m} \alpha_i (u_i - u_1) = 0$, and $\sum_{i=1}^{m} a_i = 0$.
Moreover, we have:
$\sum_{i=1}^{m} \alpha_i (u_i - u_1) = \sum_{i=1}^{m} \alpha_i u_i - (\sum_{i=1}^{m} \alpha_i) u_1 = \sum_{i=1}^{m} \alpha_i u_i$
Then:

$\sum_{i=1}^{m} \alpha_i u_i = 0$, and $\sum_{i=1}^{m} a_i = 0$.

Since the elements $a_1, a_2, \cdots, a_m \in F$ not all zero, then Y is fully dependent.

2 – Suppose that $Y = \{u_1, u_2, \cdots, u_m\}$ is another independent weak base of U. We suppose that $m \neq n$, then we can recognize two cases:

– If $m > n$, then X is fully dependent by (1), a contradiction.

– If $m < n$, then Y is fully dependent by (1), a contradiction.

Therefore, $n = m$.

3 – Suppose that $Y = \{u_1, u_2, \cdots, u_m\}$ is a finite subset of V such that $Y \not\subset U$ and $U = \langle Y \rangle_W$. Let $u \in U$, then there exist elements $\alpha_1, \alpha_2, \cdots, \alpha_m \in F$ such that $u = \sum_{i=1}^{m} \alpha_i u_i$ and $\sum_{i=1}^{m} \alpha_i = 0$. Moreover, we have:

$$u = \sum_{i=1}^{m} \alpha_i u_i = \sum_{i=1}^{m} \alpha_i u_i - (\sum_{i=1}^{m} \alpha_i) u_1 = \sum_{i=1}^{m} \alpha_i (u_i - u_1)$$

Since $u_i - u_1 \in U$ for any $1 \leq i \leq m$, then $Y_1 = \{0, u_2 - u_1, \cdots, u_m - u_1\} \subseteq U$ and $U = \langle Y_1 \rangle_W$. Thus, Y_1 contains a weak base of U by Theorem 2.14, i.e., $w \cdot dim_F U \leq m$. Therefore, $Card\, Y \geq n$.

4 – Direct by (1) and (3).

Now, we provide the following Lemma which is useful in the proof of following important Theorem:

Lemma 3.6[1]. Let V be a vector space over a field F and $X = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. The following hold:

1 – If X is weakly independent and non-independent, then $X \subset \langle X \rangle_W$, and $\langle X \rangle = \langle X \rangle_W$.

2 – If X is dependent such that $X \not\subset \langle X \rangle_W$, then X is fully dependent.

Theorem 3.7. Let V be a vector space over a field F, and U be a subspace of V. If X is a finite subset of V such that $X \not\subset U$ and $U = \langle X \rangle_W$, then X contains an independent weak base of U.

Proof. Suppose that $X = \{v_1, v_2, \cdots, v_n\}$ such that $X \not\subset U$ and $U = \langle X \rangle_W$. If X is independent, then X is an independent weak base of U. We suppose that X is not independent, then X is fully dependent by Lemma 3.6(2). Thus, there exists an element let it be $v_1 \in X$ such that $v_1 = \sum_{i=2}^{n} \alpha_i v_i$ and $\sum_{i=2}^{n} \alpha_i = 1$ where $\alpha_i \in F$ for all $2 \le i \le n$ by Lemma 2.2(6). It is obvious that $Y_1 = X \setminus \{v_1\}$ is weakly generated of U. Moreover, $Y_1 \not\subset U$ by Lemma 3.1. If Y_1 is independent, then Y_1 is an independent weak base of U. We suppose that Y_1 is not independent, then Y_1 is fully dependent by Lemma 3.6(2). Thus, there exists an element let it be $v_2 \in X$ such that $v_2 = \sum_{i=3}^{n} \beta_i v_i$ and $\sum_{i=3}^{n} \beta_i = 1$ where $\beta_i \in F$ $(3 \le i \le n)$ by Lemma 2.2(6). It's obvious that $Y_2 = X \setminus \{v_1, v_2\}$ is weakly generated of U. Moreover, $Y_2 \not\subset U$ by Lemma 3.1. If Y_2 is independent, then Y_2 is an independent weak base of U. We continue in this way to have the independent set $Y = X \setminus \{v_1, v_2, \cdots, v_m\}$ where $1 \le m < n$ which is weakly generated of U, i.e., Y is an independent weak base of U. Therefore, X contains an independent weak base of U.

Notice. Let V be a vector space over a field F, and U be a subspace of V. If X is a finite subset of V such that $X \not\subset U$ and there exists a subset $Y \subset X$ such that Y is an independent weak base of U, then it is not necessary that $U = \langle X \rangle_W$. This is shown in the following example:

Example. With R as the field of real numbers, let $U = \{(x, -x, 0); x \in R\}$ be a subspace of the vector space R^3 over R. We note that:

$X = \{(1,0,0), (0,1,0), (0,0,1)\} \not\subset U$

It is easy to show that $Y = \{(1,0,0), (0,1,0)\} \subset X$ is an independent weak base of U , but $U \neq \langle X \rangle_W$.

Theorem 3.8. Let V be a vector space over a field F and U be a subspace of V where $w \cdot \dim_F U = n$. Suppose that X is a finite subset of V consisting of n elements such that $X \not\subset U$. If $U = \langle X \rangle_W$, then X is an independent weak base of U .

Proof. Suppose that $X \not\subset U$ and $U = \langle X \rangle_W$, then X contains an independent weak base of U by Theorem 3.7 let it be S. Since $S \subseteq X$, then:

$Card\, S \leq Card\, X$.

We suppose that $Card\, S \neq Card\, X$. Since S is an independent weak base of U , then:

$w \cdot \dim_F U = Card\, S$

by Lemma 3.3. Thus, $w \cdot \dim_F U < n$, a contradiction. Hence:

$Card\, S = Card\, X$.

Since $S \subseteq X$ and X is finite, then $S = X$. Therefore, X is an independent weak base of U .

Theorem 3.9. Let V be a vector space over a field F and U be a subspace of V. Suppose that $X = \{v_1, v_2, \cdots, v_n\}$ is a weak base of U . The following hold:

1 – For every $v \in V$ such that $v \notin U$ the set:

$Y = \{v_1 + v, v_2 + v, \cdots, v_n + v\}$

is an independent weak base of U .

2 – For every finite-dimensional subspace K of V there exists a subset Z of V such that Z is an independent weak base of K .

Proof. 1 – Suppose that $X=\{v_1,v_2,\cdots,v_n\}$ is a weak base of U, then:
$w\cdot\dim_F U=n$.
Let $w\in U$, then there exist elements $\alpha_1,\alpha_2,\cdots,\alpha_n\in F$ such that:
$w=\sum_{i=1}^n \alpha_i v_i$ and $\sum_{i=1}^n \alpha_i=0$.
Let $v\in V$ such that $v\notin U$, then:
$w=\sum_{i=1}^n \alpha_i v_i+0v=\sum_{i=1}^n \alpha_i v_i+(\sum_{i=1}^n \alpha_i)v=\sum_{i=1}^n \alpha_i(v_i+v)$.
Since $\sum_{i=1}^n \alpha_i=0$, then the element w is written as a weak linear combination of the set $Y=\{v_1+v,v_2+v,\cdots,v_n+v\}$. Thus, $w\in\langle Y\rangle_W$, i.e., $U\subseteq\langle Y\rangle_W$.
On the other hand, let $u\in\langle Y\rangle_W$, then there exist elements $\beta_1,\beta_2,\cdots,\beta_n\in F$ such that:
$u=\sum_{i=1}^n \beta_i(v_i+v)$ and $\sum_{i=1}^n \beta_i=0$.
Moreover:
$u=\sum_{i=1}^n \beta_i(v_i+v)=\sum_{i=1}^n \beta_i v_i+(\sum_{i=1}^n \beta_i)v=\sum_{i=1}^n \beta_i v_i+0v=\sum_{i=1}^n \beta_i v_i$
Since $\sum_{i=1}^n \beta_i=0$, then the element u is written as a weak linear combination of X. Thus, $u\in U$, i.e., $\langle Y\rangle_W\subseteq U$. Hence, $U=\langle Y\rangle_W$.
Since $v\notin U$, then $v_i+v\notin U$ for all $1\le i\le n$, i.e., $Y\not\subset U$. Therefore, Y is an independent weak base of U by Theorem 3.8 where $Card\,Y=n$.
2 – Direct by Theorem 2.13 and (1).
Theorem 3.10. Let V be a vector space over a field F and U be a subspace of V. Suppose that $X=\{v_1,v_2,\cdots,v_n\}$ is an independent weak base of U, and $v\in V$. The following are equivalent:
1 – The set $Y=\{v_1-v,v_2-v,\cdots,v_n-v\}$ is a weak base of U.

2 – There exist unique elements $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ such that:
$v = \sum_{i=1}^{n} \alpha_i v_i$ and $\sum_{i=1}^{n} \alpha_i = 1$.
Moreover, $v \notin U$.

Proof. $(1) \Rightarrow (2)$. Suppose that $Y = \{v_1 - v, v_2 - v, \cdots, v_n - v\}$ is a weak base of U , then Y is weakly independent and non-independent, i.e., Y is dependent.
Thus, there exist elements $\beta_1, \beta_2, \cdots, \beta_n \in F$ not all zero such that:
$\sum_{i=1}^{n} \beta_i (v_i - v) = 0$.
We suppose that $\sum_{i=1}^{n} \beta_i = 0$, then $\beta_i = 0$ for all $1 \le i \le n$ because Y is weakly independent, a contradiction. Thus, $\beta = \sum_{i=1}^{n} \beta_i \ne 0$. Therefore:
$v = \sum_{i=1}^{n} (\beta^{-1} \beta_i) v_i$ and $\sum_{i=1}^{n} \beta^{-1} \beta_i = 1$.
Since X is independent, then the elements $\alpha_i = \beta^{-1} \beta_i \in F$, $1 \le i \le n$ are unique, and $v \notin U$.
$(2) \Rightarrow (1)$. Suppose that $X = \{v_1, v_2, \cdots, v_n\}$ is an independent weak base of U . Let $v = \sum_{i=1}^{n} \alpha_i v_i \in V$ where $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ such that $\sum_{i=1}^{n} \alpha_i = 1$, then:
$Y = \{v_1 - v, v_2 - v, \cdots, v_n - v\} \subseteq U$.
Let $w \in U$ then there exist elements $\beta_1, \beta_2, \cdots, \beta_n \in F$ such that $w = \sum_{i=1}^{n} \beta_i v_i$ and $\sum_{i=1}^{n} \beta_i = 0$. Moreover, we have:
$w = \sum_{i=1}^{n} \beta_i v_i - 0v = \sum_{i=1}^{n} \beta_i v_i - (\sum_{i=1}^{n} \beta_i) v = \sum_{i=1}^{n} \beta_i (v_i - v)$
Thus, $U = \langle Y \rangle_W$. Since X is an independent weak base of U , then:
$w \cdot \dim_F U = n$
by Lemma 3.3. Therefore, Y is a weak base of U by Theorem 2.15.

Theorem 3.11. Let V be a vector space over a field F and U be a subspace of V. Suppose that $Y=\{v_{i1},v_{i2},v_{i3},\cdots,v_{im}\}$ is an independent weak base of U and $v\in V$ such that $v=\sum_{j=1}^{m}\alpha_j v_{ij}$ and $\sum_{j=1}^{m}\alpha_j=1$ where $\alpha_1,\alpha_2,\cdots,\alpha_m\in F$. Then for every finite subset $X=\{v_1,v_2,\cdots,v_n\}$ of V such that $X\not\subset U$ and $Y\subseteq X$ the following are equivalent:

1 – X is weakly generated of U, i.e., $U=\langle X\rangle_W$.

2 – $X_1=\{v_1-v,v_2-v,\cdots,v_n-v\}\subseteq U$.

Proof. $(1)\Rightarrow(2)$. Since $Y\subseteq X$ and $U=\langle X\rangle_W$, then $v_i-v\in U$ for all $1\le i\le n$. Therefore, $X_1=\{v_1-v,v_2-v,\cdots,v_n-v\}\subseteq U$.

$(2)\Rightarrow(1)$. Since $Y=\{v_{i1},v_{i2},v_{i3},\cdots,v_{im}\}$ is an independent weak base of U and $v\in V$ such that $v=\sum_{j=1}^{m}\alpha_j v_{ij}$ and $\sum_{j=1}^{m}\alpha_j=1$ where $\alpha_1,\alpha_2,\cdots,\alpha_m\in F$, then:

$Y_1=\{v_{i1}-v,v_{i2}-v,\cdots,v_{im}-v\}\subseteq U$

is a weak base of U by Theorem 3.10. On the other hand, since $Y\subseteq X$, then:

$Y_1\subseteq X_1=\{v_1-v,v_2-v,\cdots,v_n-v\}\subseteq U$

Thus, $U=\langle X_1\rangle_W$ by Theorem 2.14. Since $Y\subseteq X$, then $U\subseteq\langle X\rangle_W$ by Lemma 2.1(6). Let $u\in\langle X\rangle_W$, then there exist $\beta_1,\beta_2,\cdots,\beta_n\in F$ such that $u=\sum_{i=1}^{n}\beta_i v_i$ and $\sum_{i=1}^{n}\beta_i=0$.

Moreover, we have:

$u=\sum_{i=1}^{n}\beta_i v_i=\sum_{i=1}^{n}\beta_i v_i-(\sum_{i=1}^{n}\beta_i)v=\sum_{i=1}^{n}\beta_i(v_i-v)$

Then:

$u=\sum_{i=1}^{n}\beta_i(v_i-v)$, and $\sum_{i=1}^{n}\beta_i=0$.

Hence, $u\in U$, i.e., $\langle X\rangle_W\subseteq U$. Therefore, $U=\langle X\rangle_W$.

Lemma 3.12. Let V be a vector space over a field F and U be a subspace of V. Suppose that $X=\{v_1,v_2,\cdots,v_n\}$ is a subset of V such that $X \not\subset U$ and $U=\langle X\rangle_W$. Then $v_i \notin U$ for every $1\le i\le n$.

Proof. Suppose that U is a subspace of V and $X=\{v_1,v_2,\cdots,v_n\}$ is a subset of V such that $X \not\subset U$ and $U=\langle X\rangle_W$. Then we recognize two cases:

– X is independent, then:

$v_i = 0v_1+\cdots+0v_{i-1}+1v_i+0v_{i+1}+\cdots+0v_n$

for all $v_i\in X$ where $1\le i\le n$. Therefore, every $v_i\in X$ can not be written as a weak linear combination of X where $1\le i\le n$, i.e., $v_i\notin U$.

– X is dependent, then X contains an independent weak base of U by Theorem 3.8 let it be $Y=\{v_{i1},v_{i2},\cdots,v_{im}\}$; ($m<n$). Since Y is independent, then $v_{ij}\notin U$ for every $1\le j\le m$. We suppose that, there exist an element $v_{i0}\in X$ such that $v_{i0}\notin Y$ and $v_{i0}\in U$. Since Y is an independent weak base of U, then there exist elements $a_1,a_2,\cdots,a_m\in F$ such that:

$v_{i0}=\sum_{j=1}^{m}\alpha_j v_{ij}$ and $\sum_{j=1}^{m}\alpha_j=0$

Since Y is independent, then this writing is unique. On the other hand, since $Y\subset Y_1=\{v_{i0},v_{i1},v_{i2},\cdots,v_{im}\}\subset X$, then $U=\langle Y_1\rangle_W$ by Lemma 2.1(6). Since $Card Y_1 > Card Y$, then Y_1 is fully dependent by Theorem 3.5. Hence, there exist elements $\beta_0,\beta_1,\cdots,\beta_m\in F$ not all zero such that:

$\sum_{j=0}^{m}\beta_j v_{ij}=0$ and $\sum_{j=0}^{m}\beta_j=0$.

We suppose that, $\beta_0=0$ then:

$\sum_{j=1}^{m}\beta_j v_{ij}=0$ and $\sum_{j=1}^{m}\beta_j=0$.

Since Y is independent, then $\beta_1 = \beta_2 = \cdots = \beta_m = 0$, a contradiction. Thus:

$\beta_0 \neq 0$ and $v_{i0} = \sum_{j=1}^{m} ((-\beta_0)^{-1}\beta_j)v_{ij}$.

Since $-\beta_0 = \sum_{j=1}^{m} \beta_j$, then $\sum_{j=1}^{m} (-\beta_0)^{-1}\beta_j = 1$, a contradiction, i.e., $v_{i0} \notin U$.

Therefore, $v_{i0} \notin U$ for every $1 \le i \le n$.

Theorem 3.13. Let V be a vector space over a field F and U be a subspace of V. Suppose that $X = \{v_1, v_2, \cdots, v_n\}$ is a subset of V such that $X \not\subset U$ and $U = \langle X \rangle_W$, then U is a maximal subspace of $\langle X \rangle$.

Proof. Suppose that $U = \langle X \rangle_W$, then $U \subseteq \langle X \rangle$ by Lemma 2.1(3). Since $X \not\subset U$ then $U \neq \langle X \rangle$ by Lemma 2.1(4), i.e., $U \subset \langle X \rangle$. We recognize two cases:

– X is independent, then X is a weak base of the subspace $\langle X \rangle_W$. Thus, $w \cdot dim_F U = n$ by Lemma 3.3, then $dim_F U = n-1$ by Lemma 2.11.

On the other hand, since X is independent, then $dim_F \langle X \rangle = n$. Therefore, U is a maximal subspace of $\langle X \rangle$.

– X is dependent, then there exists a subset Y of X such that Y is a base of $\langle X \rangle$, i.e., $\langle X \rangle = \langle Y \rangle$, then $\langle Y \rangle_W$ is a maximal subspace of $\langle X \rangle$ by the first case. Moreover, since $Y \subset X$ then $\langle Y \rangle_W \subset U$ by Lemma 2.1(6), then $U = \langle Y \rangle_W$. Therefore, U is a maximal subspace of $\langle X \rangle$.

Notice. Let V be a vector space over a field F, and X be a finite subset of V such that $X \not\subset \langle X \rangle_W$. If the finite subset Y of V is a base of $\langle X \rangle$, then it is not necessarily that Y is an independent weak base of $\langle X \rangle_W$. Moreover, if the finite subset Y of V is an independent weak base of $\langle X \rangle_W$, then it is not necessarily that Y is a base of $\langle X \rangle$. This is shown in the following example:

Example. With R as the field of real numbers, let $X = \{ (1,0,0), (0,1,0) \}$ be a subset of the vector space R^3 over R. Suppose that:

$V = \langle X \rangle = \{ (x, y, 0) : x, y \in R\}$

$U = \langle X \rangle_W = \{ (x, -x, 0) : x \in R\}$

It is clear that $Y = \{ (0,0,1), (1,-1,1) \}$ is an independent weak base of U, but it is not a base of V. Moreover, $Y_1 = \{ (1,1,0), (0,1,0) \}$ is a base of V, but it is not an independent weak base of U.

Theorem 3.14. Let V be a vector space over a field F and U, W are subspaces of V such that $U \subset W$. If each independent weak base of U is a base of W, then $W = V$.

Proof. Suppose that $U \subset W$ and $X = \{ v_1, v_2, \cdots, v_n \}$ is a base of U, then:

$Y = \{ 0, v_1, v_2, \cdots, v_n \}$

Is a weak base of U by Theorem 2.10. Moreover, for every $v \in V$ such that $v \notin U$, then:

$Z = \{ v, v_1 + v, v_2 + v, \cdots, v_n + v\}$

Is an independent weak base of U by Theorem 3.9. Since Z is a base of W by assumption, then $v \in W$, i.e., $V \subseteq W$. Since W is a subspace of V, then $W = V$.

Theorem 3.15. Let V be a finite-dimensional vector space over a field F and U be a subspace of V. The following are equivalent:
1 – U is a maximal subspace of V.
2 – If $X \subseteq V$ is an independent weak base of U, then X is a base of V.

Proof. $(1) \Rightarrow (2)$. Suppose that $dim_F V = n$. Since U is a maximal subspace of V, then $dim_F U = n-1$. Moreover, $w \cdot dim_F U = n$ by Lemma 2.11. Thus, there exists a finite subset $X \subset V$ such that X is an independent weak base of U by Theorem 3.9, where $Card\, X = n$ and X is independent. Therefore, X is a base of V.

$(2) \Rightarrow (1)$. Suppose that $X \subseteq V$ is an independent weak base of U, then X is a base of V by assumption, i.e., $U = \langle X \rangle_W \subseteq V = \langle X \rangle$. Therefore, U is a maximal subspace of V by Theorem 3.13.

Notice. Let V be a finite-dimensional vector space over a field F and U be a maximal subspace of V. If X is a base of V, then this does not necessarily mean that X is an independent weak base of U. This is shown in the following example:

Example. With R as the field of real numbers, let $R_3[x]$ the vector space of polynomials over R of degrees at most 3 in indeterminate x, $R_2[x]$ the vector space of polynomials over R of degrees at most 2 in indeterminate x, and $R_1[x]$ the vector space of polynomials over R of degrees at most 1 in indeterminate x. $R_1[x]$ is a subspace of $R_3[x]$, and it is a maximal subspace of $R_2[x]$.
Moreover, the set:

$$X = \{-1+x^3, 1+x^3, x+x^3\} \subseteq R_3[x]$$

Is an independent weak base of $R_1[x]$, but it is not a base of $R_2[x]$.

Theorem 3.16. Let V be a vector space over a field F, and U be a maximal subspace of V. Suppose that $X=\{v_1, v_2, \cdots, v_n\}$ is a base of V and $v=\sum_{i=1}^{n} \alpha_i v_i$ where $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ such that $\sum_{i=1}^{n} \alpha_i = 1$. The following are equivalent:
1 – X is an independent weak base of U.
2 – $Y=\{v_1-v, v_2-v, \cdots, v_n-v\}$ is a weak base of U.
Proof. 1 – Direct by Theorem 3.10.
2 – Direct by Theorem 3.9.

Theorem 3.17. Let V be a vector space over a field F, and U be a proper subspace of V. Suppose that $\Im=\{Y_i\}_{i\in I}$ is the family of all finite subsets of V such that $Y_i \not\subset U$ and $U=\langle Y_i\rangle_W$ for all $i \in I$. The following are equivalent:
1 – $U=\cap_{i\in I}\langle Y_i\rangle$.
2 – U is not a maximal subspace of V.
Proof. $(1)\Rightarrow(2)$. Let $Y_i \in \Im$. Since $Y_i \not\subset U$ and $U=\langle Y_i\rangle_W$, then Y_i contains an independent weak base of U by Theorem 3.7, let it be X_i. We suppose that U is a maximal subspace of V, then V is a finite-dimensional vector space over F and X_i is a base of V by Theorem 3.15, i.e., $V=\langle X_i\rangle$. Thus, $V=\langle Y_i\rangle$ for all $i \in I$, then $V=\cap_{i\in I}\langle Y_i\rangle$, i.e., $U=V$, a contradiction. Therefore, U is not a maximal subspace of V.
$(1)\Rightarrow(2)$. Since $Y_i \not\subset U$ and $U=\langle Y_i\rangle_W$ for every $Y_i \in \Im$, then $U \underset{\neq}{\subset} \langle Y_i\rangle$ by Lemma 2.1(3). Thus, $U \subseteq \cap_{i\in I}\langle Y_i\rangle$. Let $X \subseteq V$ is an independent weak base of U, then $X \in \Im$, and U is a maximal subspace of $\langle X\rangle$ by Theorem 3.13. Since U is not a maximal

subspace of V, then X is not a base of V by Theorem 3.15, i.e., $V \neq \langle X \rangle$. Moreover, $U \subseteq \cap_{i\in I} \langle Y_i \rangle \subseteq \langle X \rangle$.
We suppose that $U \neq \cap_{i\in I} \langle Y_i \rangle$. Since U is a maximal subspace of $\langle X \rangle$, then:
$\cap_{i\in I} \langle Y_i \rangle = \langle X \rangle$
Which means that, each independent weak base of U is a base of $\cap_{i\in I} \langle Y_i \rangle$, then $V = \cap_{i\in I} \langle Y_i \rangle$ by Theorem 3.14, i.e., $V = \langle Y_i \rangle$ for all $i \in I$.Thus, U is a maximal subspace of V, a contradiction. Therefore, $U = \cap_{i\in I} \langle Y_i \rangle$.

Theorem 3.18. Let V be a vector space over a field F, and U be a subspace of V. Suppose that $X = \{ v_1, v_2, \cdots, v_n \}$ is a subsets of V such that $X \not\subset U$ and $\langle X \rangle_W \subset U$. The following hold:
1 – $X \not\subset \langle X \rangle_W$.
2 – $v_i \notin U$ for all $1 \leq i \leq n$.
Proof. 1 – We suppose that $X \subset \langle X \rangle_W$. Since $\langle X \rangle_W \subset U$, then $X \subset U$, a contradiction. Therefore $X \not\subset \langle X \rangle_W$.
2 – Without loss of generality, we suppose that $v_1 \in U$. It is clear that:
$X_1 = \{0, v_2 - v_1, v_3 - v_1, \cdots, v_n - v_1\} \subseteq \langle X \rangle_W$.
Since $\langle X \rangle_W \subset U$, then $u_i = v_i - v_1 \in U$ for all $1 \leq i \leq n$. Thus, $v_i = u_i + v_1 \in U$ for all $1 \leq i \leq n$, i.e., $X \subset U$, a contradiction. Therefore $v_i \notin U$ for all $1 \leq i \leq n$.

Theorem 3.19. Let V be a vector space over a field F, and U be a subspace of V. Suppose that $w.dim_F\, U = n$ and

$X = \{ v_1, v_2, \cdots, v_m \}$ is an independent subsets of V such that $X \not\subset U$ where $n \geq m$. The following are equivalent:

1 – $\langle X \rangle_W \subseteq U$.

2 – X can be expanded to an independent weak base of U contained in V.

Proof. $(1) \Rightarrow (2)$. Suppose that $\langle X \rangle_W \subseteq U$. Since X is independent, then:

$X_1 = \{0, v_2 - v_1, v_3 - v_1, \cdots, v_m - v_1\} \subseteq \langle X \rangle_W$

Is a weak base of $\langle X \rangle_W$ by Theorem 3.2. Since $\langle X \rangle_W \subseteq U$ and X_1 is weakly independent, then $X_1 \subseteq U$. Moreover, by Theorem 2.16 X_1 can be expanded to a weak base of U, let it be:

$Y = \{0, v_2 - v_1, v_3 - v_1, \cdots, v_m - v_1, v_{m+1}, \cdots, v_n\}$.

Since $X \not\subset U$, then $v_1 \notin U$ by Theorem 3.18. Therefore:

$Z = \{v_1, v_2, v_3, \cdots, v_m, v_{m+1} + v_1, \cdots, v_n + v_1\}$

Is an independent weak base of U by Theorem 3.9.

$(2) \Rightarrow (1)$. Obvious.

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