العناصر الجامدة والعوادم بالنسبة لمثالي يميني

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الملخص

في هذه الورقة العلمية سندرس مفهوم العناصر الجامدة بالنسبة لمثالي يميني، حيث سنعطي توصيفاً آخر لهذه العناصر وسنورد عدداً من الأمثلة على العناصر الجامدة بالنسبة لمثالي يميني من خلال حلقة المصفوفات فوق حلقة ما وحلقة المصفوفات فوق حلقة الأعداد الصحيحة بالمقاس n.

فضلاً عن ذلك، حصلنا على الشرط اللازم والكافي كي تملك حلقة ما عناصر جامدة من خلال حلقة المصفوفات من المرتبة الثانية فوق هذه الحلقة. وفي هذا السياق أوجدنا العلاقة بين العناصر الجامدة في حلقة ما والعناصر الجامدة بالنسبة لمثالي يميني في حلقة المصفوفات فوق هذه الحلقة.

إضافة لذلك، أدخلنا مفهوم العادم لعنصر بالنسبة لمثالي يميني ما كتعميم لعادم عنصر وقد أوجدنا العديد من الشروط المكافئة لهذا المفهوم فضلاً عن العلاقة بين العوادم في حلقة ما والعوادم في حلقة المصفوفات فوق هذه الحلقة.

وقد أثبتنا أنه إذا كانت R حلقة وكان $a,b \in R$، عندئذ $b \in r(a)$ عندما وفقط عندما يوجد $x \in R$ يحقق أنه لأجل

$$\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix} \text{ فإن } \beta = \begin{bmatrix} 0 & x \\ b & 0 \end{bmatrix} \in r_P(\alpha).$$

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الكلمات المفتاحية: العناصر الجامدة، الحلقة، العناصر الجامدة بالنسبة لمثالي يميني، حلقة المصفوفات، الحلقة المحلية والمحلية بالنسبة لمثالي يميني.

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Idempotent Elements and Annihilator Relative to Right Ideal

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Abstract

In this paper we study the notion of idempotent elements relative to right ideal, we give other characterization of this elements. Also, we present several examples of idempotent elements relative to right ideal in matrices ring and ring integer modulo n.

In addition to that, new results obtained include necessary and sufficient conditions for a some ring to be has idempotent elements in term of matrices ring. Where we obtain the relationship between idempotent elements in some ring and idempotent elements relative to right ideal in matrices ring over this ring.

Finally, we introduced the concept of annihilator of element relative to right ideal as generalization of annihilator of element in some ring. Where we obtain several equivalently conditions of this concept.

In addition to that, we characterization of this concept in term matrices ring. Where we proved that if R is a ring and $a,b \in R$. Then $b \in r(a)$ if and only if there exists $x \in R$ such that for

$$\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix},\ \beta = \begin{bmatrix} 0 & x \\ b & 0 \end{bmatrix} \in r_P(\alpha).$$

Keywords: Idempotent element, Ring relative to right ideal, idempotent relative to right ideal, Ring of matrices Local ring and Local ring relative to right ideal.

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1 - Introduction.

In 1987, V. A. Andrunakievich and Yu. M. Ryabukhin [2], were the first who studied the concept of quasi-regularity and primitivity relative to right ideal. Later, V. A. Andrunakievich and A. V. Andrunakievich [1] in 1991, studied the concept of regularity of rings and idempotent elements relative to right ideals. He proved that a ring R is regular relative to right ideal $P \neq R$ if and only if for every $a \in R$, $aR + P = eR + P$ where $e \in R$ is an idempotent relative to P.

In 2011, P. Dheena and S. Manivasan [3] studied quasi-ideals of an $P-$regular near-rings. In 2018, H. Hamza studied potent rings relative to right ideal. In section 2, of this paper we study properties of idempotent element relative to right ideal and we presented example of this elements. Also, we find the relationship between idempotent elements in some ring R and idempotent elements relative to some right ideal in 2×2 matrices ring over R. In section 3, we study concept of annihilator relative to some right ideal as generalization of concept annihilator. We obtain several equivalent conditions of this concept. In addition to that, we characterization of this concept in term matrices ring.

Throughout of this paper rings R, are associative with identity. We denote the Jacobson radical of a ring R by $J(R)$

2 – Idempotents.

We start this section with the following: An element e of a ring R is called idempotent if $e^2 = e$, [4].

Definition [1]. Let R be a ring and $P \neq R$ be a right ideal of R. Recalled that an element $e \in R$ is an idempotent relative to right ideal P or ($P-$idempotent for short) if

$e^2 - e \in P$ and $eP \subseteq P$

Note that in previous definition it is easily verified that every idempotent is an $P-$idempotent. In particular, $0,1 \in R$ are $P-$idempotents. Also, if $P = 0$, then an element $e \in R$ is an $P-$idempotent if and only if e is an idempotent.

Lemma 2.1. Let R be a ring and $P \neq R$ be a right ideal of R. Then for every $P-$idempotent $e \in R$ the following hold:

1 – An element $1-e \in R$ is an $P-$idempotent.

2 – An element $e^2 \in R$ is an $P-$idempotent.

3 – An element $1-e^2 \in R$ is an $P-$idempotent.

4 – If $e \in J(R)$, then $e \in P$.

5 – If e has a right inverse, then $1-e \in P$.

6 – If $1-e$ has a right inverse, then $e \in P$.

Proof. Suppose that $e \in R$ is an $P-$idempotent, then $e^2 - e \in P$ and $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$.

1 – We have

$$(1-e)^2 = (1-e)(1-e) = 1-2e+e+p_0 = 1-e+p_0$$

so $(1-e)^2 - (1-e) = p_0 \in P$.

Also, for every $t \in P$, $(1-e)t = t - et \in P$, so $(1-e)P \subseteq P$.

2 – We have

$$(e^2)^2 = (e+p_0)(e+p_0) = e^2 + ep_0 + p_0e + p_0^2$$

So $(e^2)^2 - e^2 = ep_0 + p_0e + p_0^2 \in eP + PR + P \subseteq P$ and so $e^2P = e(eP) \subseteq eP \subseteq P$.

3 – Obvious by (1) and (2).

4 – Suppose that $e \in J(R)$, then $1-e$ has an inverse in R, so $(1-e)a = 1$ for some $a \in R$ and so $e = (e-e^2)a \in PR \subseteq P$.

5 – Suppose that e has a right inverse in R, then $ea = 1$ for some $a \in R$ and so

$$e = e^2a = (e+p_0)a = ea + p_0a = 1 + p_0a \in 1 + P$$

thus $1-e \in P$.

6 – Suppose that $1-e$ has a right inverse in R, then $(1-e)b = 1$ for some $b \in R$ and so $e = (e-e^2)b \in PR \subseteq P$.

From Lemma 2.1 and for $P = 0$ we obtain the following:

Corollary 2.2. Let R be a ring and $e \in R$ be an idempotent. Then the following statements hold:

1 – Elements e^2, $1-e$ are idempotents in R.
2 – If $e \in J(R)$, then $e = 0$.
3 – If e has a right inverse in R, then $e = 1$.
4 – If $1-e$ has a right inverse in R, then $e = 0$.

Lemma 2.3. Let R be a ring and $P \neq R$ be a right ideal of R. For every $P-$idempotent $e \in R$ the following statements hold:

1 – For every positive integer k, an element e^k is an $P-$idempotent.

2 – For every positive integer k, an element $1-e^k$ is an $P-$idempotent.

3 – For every positive integer k, an element $(1-e)^k$ is an $P-$idempotent.

Proof. 1 – Suppose that e is an $P-$idempotent, then $e^2 - e \in P$ and $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$.

Proof by induction on k. For $k = 1,2$ the assertion holds by assumption and Lemma 2.1. Suppose that e^{k-1} is an $P-$idempotent, then $(e^{k-1})^2 - e^{k-1} \in P$, $e^{k-1}P \subseteq P$ so $(e^{k-1})^2 = e^{k-1} + p_1$ for some $p_1 \in P$, thus

$$(e^k)^2 = (e^{k-1})^2 e^2 = (e^{k-1} + p_1)(e + p_0) =$$
$$= e^k + e^{k-1}p_0 + p_1 e + p_1 p_0$$

so $(e^k)^2 - e^k = p$ where $p = e^{k-1}p_0 + p_1 e + p_1 p_0 \in P$. This shows that $(e^k)^2 - e^k \in P$ and $(e^k)^2 P = ee^k P \subseteq eP \subseteq P$. Thus, e^k is an $P-$idempotent.

2 – By (1) and Lemma 2.1.
3 – By (1) and Lemma 2.1.

Next, we present an example of an $P-$idempotent elements.

Example 2.4. Let Z be the ring of integers and let $R=M_2(Z)$ be the ring of all 2×2 matrices over the ring of integers Z. It is clear that the set

$$P=\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a,b\in Z\right\}$$

is a right ideal in R and $P\neq R$. Let

$$f=\begin{bmatrix} n & n \\ 1-n & 1-n \end{bmatrix}\in R$$

where $n\in Z$, then f is an idempotent in R, so f is an $P-$idempotent in R. Let

$$e=\begin{bmatrix} n & m \\ 0 & 1 \end{bmatrix}\in R$$

where $n,m\in Z$, then e is an $P-$idempotent in R, but not idempotent, and element:

$$e^2=\begin{bmatrix} n^2 & (n+1)m \\ 0 & 1 \end{bmatrix}\in R$$

is an $P-$idempotent in R. Also, for every positive integer k the element:

$$e^k=\begin{bmatrix} n^k & m\sum_{t=0}^{k-1} n^t \\ 0 & 1 \end{bmatrix}\in R$$

is an $P-$idempotent in R. In addition to that, the element:

$$1-e=\begin{bmatrix} 1-n & -m \\ 0 & 0 \end{bmatrix}\in R$$

is an $P-$idempotent in R and

$$1-e^2=\begin{bmatrix}1-n^2 & -(n+1)m\\ 0 & 0\end{bmatrix}\in R$$

is an $P-$idempotent in R. Also, for every positive integer k the element:

$$1-e^k=\begin{bmatrix}1-n^k & -m\sum_{t=0}^{k-1}n^t\\ 0 & 0\end{bmatrix}\in R$$

is an $P-$idempotent in R. In addition to that, the element:

$$(1-e)^2=\begin{bmatrix}(1-n)^2 & -(1-n)m\\ 0 & 0\end{bmatrix}\in R$$

is an $P-$idempotent in R. Also, for every positive integer k the element:

$$(1-e)^k=\begin{bmatrix}(1-n)^k & -m(1-n)^{k-1}\\ 0 & 0\end{bmatrix}\in R$$

is an $P-$idempotent in R.

Let R be a ring and $P\neq R$ be a right ideal of R. Let $Pid(R)$ the set of all an $P-$idempotent elements in R. It is clear that $Pid(R)$ is a nonempty subset of R, because $0,1\in Pid(R)$.

For every $f,e\in Pid(R)$, we define the relation $(\sim)$ on $Pid(R)$ as following:

$$e\sim f \quad\Leftrightarrow\quad e-f\in P$$

It is easy to see that $(\sim)$ is an equivalent relation on $Pid(R)$. If $e\in Pid(R)$, then the equivalent class $\bar{e}=\{f : f\in Pid(R); e-f\in P\}$.

Lemma 2.5. Let R be a ring and $P\neq R$ be a right ideal in R. If $e,g\in R$ such that $e-g\in P$, then g is an $P-$idempotent if and only if e is an $P-$idempotent.

Proof. Suppose that $e-g\in P$, then $e=g+p_1$ for some $p_1\in P$. Assume that g is an $P-$idempotent, then $g^2-g\in P$, $gP\subseteq P$. So $g^2=g+p_0$ for some $p_0\in P$ and

$$e^2=(g+p_1)(g+p_1)=g^2+gp_1+p_1g+p_1p_1=$$
$$=g+p_0+gp_1+p_1g+p_1p_1=$$
$$=g+p_1+(-p_1+p_0+gp_1+p_1g+p_1p_1)$$

For $p'=p_1+(-p_1+p_0+gp_1+p_1g+p_1p_1)\in P$, we have $e^2-e=p'\in P$ and

$$eP\subseteq gP+p_1\subseteq P$$

This shows that e is an $P-$idempotent. Similarly, we can prove the converse.

Lemma 2.6. Let R be a ring and $P\neq R$ be a right ideal of R. Then for every $P-$idempotent $e\in R$ the following hold:

1 – For every positive integer k, $e^k\sim e$, i.e., $e^k-e\in P$.

2 – For every positive integer k, $(1-e)^k\sim(1-e)$, i.e., $(1-e)^k-(1-e)\in P$.

3 –For every positive integer k, $1-e^k\sim 1-e$, i.e., $(1-e^k)-(1-e)\in P$.

Proof. 1 – Assume that $e\in R$ is an $P-$idempotent, then $e^2-e\in P$ and $eP\subseteq P$, so $e^2-e=p_0$ for some $p_0\in P$. Let k be a positive integer, proof by induction on k.

For $k=1,2$ the assertion holds by definition. Suppose that $e^{k-1}\sim e$, then $e^{k-1}-e\in P$ so $e^{k-1}=ep_1$ for some $p_1\in P$. Thus,

$$e^k-e=ee^{k-1}-e=e(e+p_1)-e=(e^2-e)+ep_1=$$
$$=p_0+ep_1\in P+eP\subseteq P$$

so $e^k \sim e$.

2 – It is clear by (1), because $1-e$ is an $P-$idempotent.

3 – $(1-e^k)-(1-e) = -e^k + e = -(e^k - e) \in P$ by (1).

In example 2.4 we found that the element:

$$e = \begin{bmatrix} n & m \\ 0 & 1 \end{bmatrix} \in R = M_2(Z)$$

for every $n, m \in Z$, is an $P-$idempotent. So by Lemma 2.6 for every positive integer k, $e^k \sim e$, i.e.,

$$e = \begin{bmatrix} n & m \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} n^k & m\sum_{t=0}^{k-1} n^t \\ 0 & 1 \end{bmatrix} = e^k$$

for every $n, m \in Z$ and $(1-e)^k \sim 1-e$, i.e.,

$$1-e = \begin{bmatrix} 1-n & -m \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} (1-n)^k & -m(1-n)^{k-1} \\ 0 & 0 \end{bmatrix} = (1-e)^k$$

also, $1-e^k \sim 1-e$, i.e.,

$$1-e^k = \begin{bmatrix} 1-n^k & -m\sum_{t=0}^{k-1} n^t \\ 0 & 0 \end{bmatrix} \in \overline{1-e}$$

$$1-e = \begin{bmatrix} 1-n & -m \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1-n^k & -m\sum_{t=0}^{k-1} n^t \\ 0 & 0 \end{bmatrix} = 1-e^k$$

Lemma 2.7. Let R be a ring and $P \neq R$ be a right ideal of R. Then for every $P-$idempotent $e \in R$ the following hold:

1 – If for some positive integer k, e^k has a right inverse, then $1-e \in P$.

2 – If for some positive integer k, $(1-e)^k$ has a right inverse, then $e \in P$.

Proof. Suppose that $e \in R$ is an $P-$idempotent, then $e^2 - e \in P$ and $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$.

1 – Assume that e^k has a right inverse for some positive integer k, since by Lemma 2.3 e^k is an $P-$idempotent, $1 - e^k \in P$ by Lemma 2.3, so $1 = e^k + p_1$ for some $p_1 \in P$.

On the other hand, since by Lemma 2.6 $e^k \sim e$, $e^k - e \in P$, so $e^k = e + p_2$ for some $p_2 \in P$ therefore $1 = e^k + p_1 = e + p_2 + p_1$, thus $1 - e = p_2 + p_1 \in P$.

2 – Suppose that $(1-e)^k$ has a right inverse for some positive integer k, since by Lemma 2.3 $(1-e)^k$ is an $P-$idempotent, by (1) $1-(1-e) \in P$, so $e \in P$.

Definition. Let R be a ring and $P \neq R$ be a right ideal of R. Two $P-$idempotents $e, f \in R$ are called $P-$orthogonal idempotents if $ef \in P$. It is clear that if $e \in R$ is an $P-$idempotent, then $e, (1-e)$ are $P-$orthogonal idempotents.

Lemma 2.8. Let R be a ring and $P \neq R$ be a right ideal of R. Then for every $P-$idempotent $e \in R$ the following hold:

1 – For every positive integer k, $e^k, 1-e$ are $P-$orthogonal idempotents.

2 – For every positive integer k, $(1-e)^k, e$ are $P-$orthogonal idempotents.

3 – For every positive integer k, $1-e^k, e$ are $P-$orthogonal idempotents.

Proof. (1). Suppose that $e \in R$ is an $P-$idempotent, then by Lemma 2.3 e^k is and $P-$idempotent for every positive integer k. Proof by induction on k. For $k=1$ the assertion holds by definition. Suppose that $e^{k-1}(1-e)\in P$, then

$$e^k(1-e)=ee^{k-1}(1-e)\in eP\subseteq P$$

so e^k and $1-e$ are $P-$orthogonal idempotents.

2 – Since $e\in R$ is an $P-$idempotent, then by Lemma 2.3 $(1-e)^k$ is an $P-$idempotent for every positive integer k. Proof by induction on k. For $k=1$ the assertion holds by definition. Suppose that $e(1-e)^{k-1}\in P$, then

$$e(1-e)^k=e(1-e)^{k-1}(1-e)\in PR\subseteq P$$

so e and $(1-e)^k$ are $P-$orthogonal idempotents.

3 –Since $e\in R$ is an $P-$idempotent, then by Lemma 2.3 $(1-e)^k$ is an $P-$idempotent for every positive integer k. On the other hand, $e^2-e\in P$ and $eP\subseteq P$, so $e^2=e+p_0$ for some $p_0\in P$. Proof by induction on k. For $k=1$ the assertion holds by definition. Suppose that $e(1-e^{k-1})\in P$, then

$$e(1-e^k)=e-e^{k-1}=e-e^{k-1}(e+p_0)=$$

$$=e-e^k+e^{k-1}p_0=e(1-e^{k-1})+e^{k-1}p_0\in P+e^{k-1}P\subseteq P$$

so e and $1-e^k$ are $P-$orthogonal idempotents.

Lemma 2.9. Let R be a ring and $P\neq R$ be a right ideal in R. Suppose that $e,f\in Pid(R)$. If e,f are $P-$orthogonal idempotents, then e,h are $P-$orthogonal idempotents for every $h\in\overline{f}$.

Proof. Suppose that e,f are $P-$orthogonal idempotents, then $ef\in P$, so $ef=p_0$ for some $p_0\in P$. Let $h\in\overline{f}$, then $h-f\in P$, so $h=f+p_1$ for some $p_1\in P$, thus

$$eh=e(f+p_1)=ef+ep_1=p_0+ep_1\in P+eP\subseteq P$$

so e, h are $P-$orthogonal idempotents.

Example 2.10. In Example 2.4 we found that the element:

$$e = \begin{bmatrix} n & m \\ 0 & 1 \end{bmatrix} \in R = M_2(Z)$$ and

$$1-e = \begin{bmatrix} 1-n & -m \\ 0 & 0 \end{bmatrix} \in R = M_2(Z)$$

for every $n, m \in Z$, are $P-$idempotents. So by Lemma 2.3 for every positive integer $k \in Z$

$$e^k = \begin{bmatrix} n^k & m\sum_{t=0}^{k-1} n^t \\ 0 & 1 \end{bmatrix} \in R = M_2(Z)$$ and

$$1-e = \begin{bmatrix} 1-n & -m \\ 0 & 0 \end{bmatrix} \in R = M_2(Z)$$

are $P-$orthogonal idempotents. Also,

$$e = \begin{bmatrix} n & m \\ 0 & 1 \end{bmatrix} \in R = M_2(Z)$$ and

$$(1-e)^k = \begin{bmatrix} (1-n)^k & -m(1-n)^{k-1} \\ 0 & 0 \end{bmatrix} \in R = M_2(Z)$$

are $P-$orthogonal idempotents.

Next, we present another example of idempotent relative to right ideal:

Lemma 2.11. Let Z be the ring of integers and let $R = M_2(Z)$ be the ring of all 2×2 matrices over the ring of integers Z. It is clear that the set

$$Q = \left\{ \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} : a, b \in Z \right\}$$

is a right ideal in R and $P \neq R$. Let

$$e = \begin{bmatrix} 1 & 0 \\ n & m \end{bmatrix} \in R = M_2(Z)$$

where $n, m \in Z$, then e is a $Q-$idempotent in R. Also, the element:

$$e^2 = \begin{bmatrix} 1 & 0 \\ n(1+m) & m^2 \end{bmatrix} \in R = M_2(Z)$$

is a $Q-$idempotent in R and for every positive integer $k \in Z$, the element:

$$e^k = \begin{bmatrix} 1 & 0 \\ n\sum_{t=0}^{k-1} m^t & m^k \end{bmatrix} \in R = M_2(Z)$$

is a $Q-$idempotent in R, and the element:

$$1-e = \begin{bmatrix} 0 & 0 \\ -n & 1-m \end{bmatrix} \in R = M_2(Z)$$

is a $Q-$idempotent in R, also the element:

$$1-e^2 = \begin{bmatrix} 0 & 0 \\ -n(1+m) & 1-m^2 \end{bmatrix} \in R = M_2(Z)$$

is a $Q-$idempotent in R, and the element:

$$1-e^k = \begin{bmatrix} 0 & 0 \\ -n\sum_{t=0}^{k-1} m^t & 1-m^k \end{bmatrix} \in R = M_2(Z)$$

is a $Q-$idempotent in R, and the element:

$$(1-e)^k = \begin{bmatrix} 0 & 0 \\ -n(1-m)^{K-1} & (1-m)^k \end{bmatrix} \in R = M_2(Z)$$

is a $Q-$idempotent.

Note that $\bar{e}=\left\{\begin{bmatrix}1 & 0\\ x & y\end{bmatrix}: \text{for all } x,y\in Z\right\}$ so $e\sim e^2\sim e^3\sim\cdots\sim e^k$.

Also, $\overline{1-e}=\left\{\begin{bmatrix}0 & 0\\ x & y\end{bmatrix}: \text{for all } x,y\in Z\right\}$ so $1-e\sim 1-e^2\sim 1-e^3\sim\cdots\sim 1-e^k$.

In addition to that, $f=\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}$, $h=\begin{bmatrix}1 & 0\\ 3 & 4\end{bmatrix}\in R=M_2(Z)$ are not $Q-$orthogonal idempotents.

We again use the notation, let R be a ring and $S=M_2(R)$ be the ring of all 2×2 matrices over a ring R. It is clear that the sets:

$$P=\left\{\begin{bmatrix}a & b\\ 0 & 0\end{bmatrix}: a,b\in R\right\} \quad \text{and} \quad Q=\left\{\begin{bmatrix}0 & 0\\ a & b\end{bmatrix}: a,b\in R\right\}$$

are right ideals in S such that $P\neq S$ and $Q\neq S$.

The connection between the idempotent elements in R and $P-$idempotent elements ($Q-$idempotent elements) in S we provide in the following:

Theorem 2.12. For any element $e\in R$ the following hold:

1 – If e is an idempotent in R, then for every $x,y\in R$, the element

$$e_0=\begin{bmatrix}x & y\\ 0 & e\end{bmatrix}\in S$$

is an $P-$idempotent in S.

2 – If for some $x,y\in R$, an element $e_0=\begin{bmatrix}x & y\\ 0 & e\end{bmatrix}\in S$ is an $P-$idempotent in S, then e is an idempotent in R.

3 – If e is an idempotent in R, then for every $x, y \in R$, the element $e_0 = \begin{bmatrix} e & 0 \\ x & y \end{bmatrix} \in S$ is an $Q-$idempotent in S.

4 – If for some $x, y \in R$, the element $e_0 = \begin{bmatrix} e & 0 \\ x & y \end{bmatrix} \in S$ is an $Q-$idempotent in S, then e is an idempotent in R.

Proof. 1 – Suppose that e is an idempotent in R, then for every $x, y \in R$,

$$e_0^2 - e_0 = \begin{bmatrix} x^2 & xy+ye \\ 0 & e^2 \end{bmatrix} - \begin{bmatrix} x & y \\ 0 & e \end{bmatrix} = \begin{bmatrix} x^2-x & xy+ye-y \\ 0 & e^2-e \end{bmatrix} \in P$$

and for every $p = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in P$, where $a, b \in R$, $e_0 p = \begin{bmatrix} xa & xb \\ 0 & 0 \end{bmatrix} \in P$, so $e_0 P \subseteq P$, this shows that e_0 is an $P-$idempotent in S.

2 – Let $x, y \in R$ such that $e_0 = \begin{bmatrix} x & y \\ 0 & e \end{bmatrix} \in S$ is an $P-$idempotent in S. Since $e_0^2 - e_0 \in P$,

$$\begin{bmatrix} x^2 & xy+ye \\ 0 & e^2 \end{bmatrix} - \begin{bmatrix} x & y \\ 0 & e \end{bmatrix} = \begin{bmatrix} x^2-x & xy+ye-y \\ 0 & e^2-e \end{bmatrix} = \begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix}$$

for some $a', b' \in R$, so $e^2 = e$. This shows that e is an idempotent in R.

3 – Similarly as in (1).

4 – Similarly as in (2).

Example 2.13. Let Z_6 be the ring of integer modulo 6. It is known that $3,4 \in Z_6$ are idempotent elements in Z_6. Let $R = M_2(Z_6)$ be the ring of all 2×2 matrices over a ring Z_6. It is clear that the set:

$$P = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in R \right\}$$

is a right ideal in R such that $P \neq R$. For every $x, y \in Z_6$, elements:

$\begin{bmatrix} x & y \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} x & y \\ 0 & 4 \end{bmatrix}$

are $P-$idempotents in R.

3 – Annihilator Relative to Right Ideal.

In this section we study the concept of annihilator relative to right ideal, we start with the following:

Definition. Let R be a ring, $P \neq R$ be a right ideal of R and $a \in R$. We called the set

$r_P(a) = \{x : x \in R; ax \in P\}$

the right annihilator of a in R relative to right ideal P, or $P-$annihilator for short, of a in R. Properties of $P-$annihilator we study in the following:

Lemma 3.1 [5]. Let R be a ring and $P \neq R$ be a right ideal of R. Then the following statements hold:

1 – For every $a \in R$ the set $r_P(a)$ is a right ideal in R.

2 – For every $a \in R$, $aP \subseteq P$ if and only if $P \subseteq r_P(a)$.

3 – For every $a \in R$, $r_P(a) = R$ if and only if $a \in P$.

4 – $r_P(1) = P$.

5 – For every $P-$idempotent $e \in R$, $P \subseteq r_P(e)$.

Lemma 3.2. Let R be a ring and $P \neq R$ be a right ideal in R. For every $a, b \in R$ such that $a - b \in P$, $r_P(a) = r_P(b)$.

Proof. Suppose that $a-b \in P$, then $a = b + p_0$ for some $p_0 \in P$. Let $x \in r_P(a)$, then $ax \in P$, so $bx = ax + p_0 x \in P + PR \subseteq P$ and so $x \in r_P(b)$. Thus, $r_P(a) \subseteq r_P(b)$.

Similarly, we can prove that $r_P(b) \subseteq r_P(a)$.

Let R be a ring and $P \neq R$ be a right ideal in R. For every $a, b \in R$, we define the relation ($\sim$) on R as following:

$a \sim b \quad \Leftrightarrow \quad a-b \in P$

it is clear that ($\sim$) is an equivalent relation on R. If $a \in R$, then the equivalent class

$\overline{a} = \{b : b \in R; a \sim b\} = \{b : b \in R; a-b \in P\}$

and by Lemma 3.2 $\overline{a} = \{b : b \in R; r_P(a) = r_P(b)\}$. So the set $\overline{R} = \{ r_P(a) : a \in R\}$ constitute a partition of a ring R.

Lemma 3.3. Let R be a ring and $P \neq R$ be a right ideal of R. Then for every $P-$idempotent $e \in R$ the following statements hold:

1 – $r_P(e) = (1-e)R + P$.

2 – $r_P(1-e) = eR + P$.

3 – $r_P(e^2) = r_P(e)$ and $r_P((1-e)^2) = r_P(1-e)$.

Proof. Suppose that $e \in R$ is an $P-$idempotent, then $e^2 - e \in P$, $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$.

1 – Let $x \in r_P(e)$, then $ex \in P$ and $x = ex + (1-e)x \in (1-e)R + P$ thus $r_P(e) \subseteq (1-e)R + P$.

Let $y \in (1-e)R + P$, then $y = (1-e)z + p_1$ where $z \in R$, $p_1 \in P$, so

$ey = (e-e^2)z + ep_1 \in PR + eP \subseteq P$

This shows that $r_P(1-e) = eR + P$.

2 – Since $e \in R$ is an $P-$idempotent, then $1-e \in R$ is an $P-$idempotent and by (1),
$r_P(1-e)=(1-(1-e))R+P=eR+P$.

3 – Let $x \in r_P(e^2)$, then $e^2x \in P$, so $e^2x = p_2$ for some $p_2 \in P$ and so $(e+p_0)x = p_2$
therefore $ex = -p_0x + p_2 \in PR + P \subseteq P$, thus $x \in r_P(e)$,i.e. $r_P(e^2) \subseteq r_P(e)$.

Let $y \in r_P(e)$, then $ey \in P$ and $e^2y \in eP \subseteq P$ so $y \in r_P(e^2)$,i.e. $r_P(e) \subseteq r_P(e^2)$, thus
$r_P(e^2) = r_P(e)$. Also, $r_P((1-e)^2) = r_P(1-e)$, hence $1-e$ is an $P-$idempotent.

Note that in Lemma 3.3 and for $P=0$ we derive the following:

Corollary 3.4. Let R be a ring and $e \in R$ be an idempotent. Then the following hold:
$r(e)=(1-e)R$, $\quad r(1-e)=eR$

Lemma 3.5. Let R be a ring and $P \neq R$ be a right ideal in R. If $e \in R$ is an $P-$idempotent, then the following hold:

1 – For every positive integer k, $r_P(e) = r_P(e^k)$.

2 – For every positive integer k, $r_P(1-e) = r_P((1-e)^k)$.

3 – For every positive integer k, $r_P(1-e) = r_P(1-e^k)$.

Proof. 1 – Suppose that $e \in R$ is an $P-$idempotent and k is a positive integer. Proof by induction on k. For $k=1$ the assertion holds by definition. For $k=2$,
$r_P(e^2) = r_P(e)$ by Lemma 2.3. Suppose that $r_P(e^{k-1}) = r_P(e)$. It is clear that $r_P(e) \subseteq r_P(e^2)$, because if $x \in r_P(e)$, then $ex \in P$, so $e^kx = e^{k-1}ex \in e^{k-1}P \subseteq P$ and so $x \in r_P(e^{k-1})$.

Let $y \in r_P(e^k)$, then $e^k y \in P$ and $e^{k-1}ey \in P$, so $ey \in r_P(e^{k-1}) = r_P(e)$, this shows that $e^2 y \in P$, i.e., $y \in r_P(e^2) = r_P(e)$, thus $r_P(e^k) \subseteq r_P(e)$.

2 – Implies by (1), hence $1-e \in R$ is an $P-$idempotent.

3 – We have $1-e$ and $1-e^k$ are $P-$idempotents for every positive integer k by Lemma 2.1 and Lemma 2.3.

Proof by induction on k. For $k=1$ the assertion holds by definition. For $k=2$, since e is an $P-$idempotent, $e^2-e \in P$ and $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$. Let $x \in r_P(1-e)$, then $(1-e)x \in P$, so $x = ex + p_1$ for some $p_1 \in P$ and so

$$x = (e^2 - p_0)x + p_1 = e^2 x - p_0 x + p_1$$

$$(1-e^2)x = -p_0 x + p_1 \in PR + P \subseteq P$$

so $x \in r_P(1-e^2)$ and so $r_P(1-e) \subseteq r_P(1-e^2)$. Let $y \in r_P(1-e^2)$, then $(1-e^2)y \in P$ and

$y = e^2 y + p_3$ for some $p_3 \in P$ and so

$y = (e + p_0)y + p_3 = ey + p_0 y + p_3$ and

$$(1-e)y = p_0 y + p_3 \in PR + P \subseteq P$$

so $y \in r_P(1-e)$ and $r_P(1-e^2) \subseteq r_P(1-e)$.

Assume that $r_P(1-e) = r_P(1-e^{k-1})$. Let $x \in r_P(1-e)$, then $x \in r_P(1-e^{k-1})$, so $x = e^{k-1}x + p'$ for some $p' \in P$ and

$$x = e^{k-2}ex + p' = e^{k-2}(e^2 + p_0)x + p' = e^k x - e^{k-2} p_0 x + p'$$

$$(1-e^{k-1})x = -e^{k-2} p_0 x + p' \in e^{k-1}PR + P \subseteq eP + P \subseteq P$$

so $x \in r_P(1-e^k)$ and $r_P(1-e) \subseteq r_P(1-e^k)$.

Let $y \in r_P(1-e^k)$, then $y = e^k y + p''$ for some $p'' \in P$,

$$y = e^{k-2}e^2 y + p'' = e^{k-2}(e + p_0)y + p'' = e^{k-1}y + e^{k-2}p_0 y + p''$$

$$(1-e^{k-1})y = e^{k-2}p_0 y + p'' \in e^{k-2}PR + P \subseteq eP + P \subseteq P$$

so $y \in r_P(1-e^{k-1}) = r_P(1-e)$, thus $r_P(1-e^k) \subseteq r_P(1-e)$.

Example 3.6. Let Z be the ring of integers and let $R = M_2(Z)$ be the ring of all 2×2 matrices over the ring of integers Z. It is clear that the set:

$$P = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in Z \right\}$$

is a right ideal in R and $P \neq R$. Let $a = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \in R$ then:

$$r_P(a) = \left\{ \begin{bmatrix} x & y \\ -3x & -3y \end{bmatrix} : x, y \in Z \right\}$$

So for $a = \begin{bmatrix} 1 & n \\ m & 1 \end{bmatrix} \in R$ where $n, m \in Z$, then

$$r_P(a) = \left\{ \begin{bmatrix} x & y \\ -mx & -my \end{bmatrix} : x, y \in Z \right\}.$$

Let $b = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \in R$, then $r_P(b) = \left\{ \begin{bmatrix} -3x & -3y \\ x & y \end{bmatrix} : x, y \in Z \right\}$,

so for $a = \begin{bmatrix} n & 1 \\ 1 & m \end{bmatrix} \in R$

where $n, m \in Z$, then $r_P(b) = \left\{ \begin{bmatrix} -mx & -my \\ x & y \end{bmatrix} : x, y \in Z \right\}$.

Let $c=\begin{bmatrix}2 & 3\\ 4 & 1\end{bmatrix}\in R$ then $r_P(c)=\left\{\begin{bmatrix}x & y\\ -4x & -4y\end{bmatrix}: x,y\in Z\right\}$. So for $d=\begin{bmatrix}n & m\\ s & 1\end{bmatrix}\in R$ where $n,m,s\in Z$, then $r_P(d)=\left\{\begin{bmatrix}x & y\\ -sx & -sy\end{bmatrix}: x,y\in Z\right\}$.

We again use the notation, let R be a ring and $S=M_2(R)$ be the ring of all 2×2 matrices over a ring R. It is clear that the subsets:

$P=\left\{\begin{bmatrix}a & b\\ 0 & 0\end{bmatrix}: a,b\in R\right\}$ and $Q=\left\{\begin{bmatrix}0 & 0\\ a & b\end{bmatrix}: a,b\in R\right\}$

are right ideals in S such that $P\neq S$ and $Q\neq S$. The connection between the right annihilator in R and $P-$annihilator ($Q-$annihilator) in S we provide in the following:

Theorem 3.7. For every ring R the following hold:

1 – If $\alpha=\begin{bmatrix}u & v\\ 0 & a\end{bmatrix}\in S$ where $u,v,a\in R$, then $\beta=\begin{bmatrix}x & y\\ b & 0\end{bmatrix}\in r_P(\alpha)$ for every $x,y\in R$ and for every $b\in r(a)$.

2 – If $\alpha=\begin{bmatrix}u & v\\ 0 & a\end{bmatrix}\in S$ where $u,v,a\in R$, then $\gamma=\begin{bmatrix}x & y\\ 0 & b\end{bmatrix}\in r_P(\alpha)$ for every $x,y\in R$ and for every $b\in r(a)$.

3 – If $\alpha = \begin{bmatrix} a & 0 \\ u & v \end{bmatrix} \in S$ where $u, v, a \in R$, then $\beta = \begin{bmatrix} b & 0 \\ x & y \end{bmatrix} \in r_Q(\alpha)$ for every $x, y \in R$ and for every $b \in r(a)$.

4 – If $\alpha = \begin{bmatrix} a & 0 \\ u & v \end{bmatrix} \in S$ where $u, v, a \in R$, then $\gamma = \begin{bmatrix} 0 & b \\ x & y \end{bmatrix} \in r_Q(\alpha)$ for every $x, y \in R$ and for every $b \in r(a)$.

Proof. 1 – Let $x, y \in R$ and $b \in r(a)$, then $ab = 0$ and

$$\alpha\beta = \begin{bmatrix} u & v \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} x & y \\ b & 0 \end{bmatrix} = \begin{bmatrix} ux + vb & uy \\ ab & 0 \end{bmatrix} \in P$$

so $\beta \in r_P(\alpha)$.

2 – Let $x, y \in R$ and $b \in r(a)$, then $ab = 0$ and

$$\alpha\gamma = \begin{bmatrix} u & v \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} x & y \\ 0 & b \end{bmatrix} = \begin{bmatrix} ux & uy + vb \\ 0 & ab \end{bmatrix} \in P$$

so $\gamma \in r_P(\alpha)$. (3) Similarly as in (1) and (4) Similarly as in (2).

Theorem 3.8. For every ring R the following hold:

1 – Let $\alpha = \begin{bmatrix} u & v \\ 0 & a \end{bmatrix} \in S$ where $u, v, a \in R$, if $\beta = \begin{bmatrix} x & y \\ b & 0 \end{bmatrix} \in r_P(\alpha)$ for $x, y, b \in R$ then $b \in r(a)$.

2 – Let $\alpha=\begin{bmatrix} u & v \\ 0 & a \end{bmatrix} \in S$ where $u,v,a \in R$, if $\gamma=\begin{bmatrix} x & y \\ 0 & b \end{bmatrix} \in r_P(\alpha)$ for $x,y,b \in R$ then $b \in r(a)$.

3 – Let $\alpha=\begin{bmatrix} a & 0 \\ u & v \end{bmatrix} \in S$ where $u,v,a \in R$, if $\beta=\begin{bmatrix} b & 0 \\ x & y \end{bmatrix} \in r_Q(\alpha)$ for $x,y,b \in R$ then $b \in r(a)$.

4 – Let $\alpha=\begin{bmatrix} a & 0 \\ u & v \end{bmatrix} \in S$ where $u,v,a \in R$, if $\gamma=\begin{bmatrix} 0 & b \\ x & y \end{bmatrix} \in r_Q(\alpha)$ for $x,y,b \in R$ then $b \in r(a)$.

Proof. 1 – Suppose that $\beta=\begin{bmatrix} x & y \\ b & 0 \end{bmatrix} \in r_P(\alpha)$, then

$$\alpha\beta=\begin{bmatrix} u & v \\ 0 & a \end{bmatrix}\cdot\begin{bmatrix} x & y \\ b & 0 \end{bmatrix}=\begin{bmatrix} ux+vb & uy \\ ab & 0 \end{bmatrix} \in P$$

so $ab=0$ and so $b \in r(a)$.

2 – Suppose that $\gamma=\begin{bmatrix} x & y \\ 0 & b \end{bmatrix} \in r_P(\alpha)$, then

$$\alpha\gamma=\begin{bmatrix} u & v \\ 0 & a \end{bmatrix}\cdot\begin{bmatrix} x & y \\ 0 & b \end{bmatrix}=\begin{bmatrix} ux & uy+vb \\ 0 & ab \end{bmatrix} \in P$$

so $ab=0$ and so $b \in r(a)$. (3) Similarly as in (1). (4) Similarly as in (2).

From Theorem 3.7 and Theorem 3.8 we obtain the following:

Corollary 3.9. Let R be a ring and $a,b \in R$. Then the following hold:

1 – $b \in r(a)$ if and only if there exists $x \in R$ such that for

$$\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix} \text{ and } \beta = \begin{bmatrix} 0 & x \\ b & 0 \end{bmatrix}$$

$\beta \in r_P(\alpha)$.

2 – $b \in r(a)$ if and only if there exists $x \in R$ such that for

$$\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix} \text{ and } \gamma = \begin{bmatrix} x & 0 \\ 0 & b \end{bmatrix}$$

$\gamma \in r_P(\alpha)$.

3 – $b \in r(a)$ if and only if there exists $x \in R$ such that for

$$\alpha = \begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix} \text{ and } \beta = \begin{bmatrix} b & 0 \\ 0 & x \end{bmatrix}$$

$\beta \in r_Q(\alpha)$.

4 – $b \in r(a)$ if and only if there exists $x \in R$ such that for

$$\alpha = \begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix} \text{ and } \gamma = \begin{bmatrix} 0 & b \\ x & 0 \end{bmatrix}$$

$\gamma \in r_Q(\alpha)$.

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